4000 Years of Algebra

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Last time I covered over 4000 years of geometry in one hour, from the Egyptian pyramids to Penrose tilings. Today I'll attempt to do the same for algebra.

Inevitably this lecture will be less pictorial and more technical than last time. It will also be less wide-ranging - in fact, much of it comes under the single heading of 'how to solve equations'. Some of it relates to various types of number, a topic that I'll be dealing with in my next lecture.

Egypt

Once again, we'll start with the Egyptians. Much of what we know about Egyptian mathematics comes from a handful of papyruses - in particular, the Rhind papyrus in the British Museum. Much of this papyrus consists of about 80 problems, used apparently for the training of scribes.

A few problems are what we now think of as algebraic, such as the following, number 25 on the Rhind papyrus. A quantity and its half added together become 16. What is the quantity? In modern algebraic terminology, which of course they didn't use, we'd be trying to solve the equation $x + \frac{1}{2}x = 16$.

Their approach was to use the method of false position, in which they guessed a convenient solution and then scaled it up or down as necessary. Here, it's convenient to try 2, so that the quantity and its half come to 3. But we want 16, so we scale the 3 up to 16 and apply the same scaling to 2 to obtain the answer.

To carry out the calculation, they used doubling, halving and multiplication by 2/3. The appropriate rows are then singled out, and we obtain the answer, 10 2/3. Finally, under the heading of 'Do it thus', they checked their solution.

This mention of - leads us to Egyptian fractions, which are very different from ours. Apart from 2/3, all their fractions were made up from unit fractions, or reciprocals 1/n. So, where we'd write 2/11 they'd write 1/61/66, and instead of 2/13 they'd write 1/8 1/52 1/104.

Their ability to calculate with these unit fractions is remarkable and can be seen in Problem 31 of the Rhind papyrus. A quantity, its 2/3, its ½, and its 1/7, added together become 33. What is the quantity? In our algebraic notation, this problem requires us to solve the equation $x + 2/3x + \frac{1}{2}x + 1/7x = 33$. Their answer, which we'd write as 1428/97, they gave as 14 ¼ 1/56 1/97 1/194 1/388 1/679 1/776 - a highly impressive feat of calculation.

How did they do it? They used extensive tables of numbers, breaking the fraction down to a succession of fractions of the form 2/n and then combining these repeatedly. To this end, the Rhind papyrus includes a table of fractions of the form 2/n, for all odd numbers n from 5 up to 101.

Mesopotamian mathematics

Let's now turn our attention to Mesopotamian mathematics. Although dating from around the same time as the Rhind papyrus, the tablets we'll look at are very different in content. Using a wedge-shaped stylus, the Mesopotamians imprinted their symbols in moist clay and left the tablet to dry in the sun. Unlike the Egyptians' decimal counting system, the Mesopotamian system was based on 60 - remnants survive in our measurement of time (60 seconds in a minute, 60 minutes in an hour).

There were essentially two types of mathematical tablet - table texts, with tables of numbers for use in calculations, and problem texts, in which problems are posed and solved.

One question from a problem text leads to what we now call a linear equation:

I found a stone, but did not weigh it; after I weighed out 6 times its weight, added 2 gin (a unit of weight), and added one-third of one-seventh multiplied by 24, I weighed it: 1 ma-na (another unit of weight). What was the weight of the stone?

This problem clearly isn't a practical one - if we want the weight of the stone, why don't we just weigh it in the first place? It is just one of 23 such problems, all on the same tablet and all ending up with 1 ma-na, which leads us to believe that the tablet was a teaching tablet. Given that 1 ma-na equals 60 gin, we argue as follows, using modern algebraic notation:

if x is the weight of the stone, then (6x + 2) + 1/3 . 1/7 . 24 (6x + 2) = 60 gin, so x = 41/3 gin.

Note that we take 1/3 of 1/7 times 24 not of x, but of 6x + 2, the stage that we've just reached in the calculation.

The next problem text is more complicated.

I have subtracted the side of my square from the area: 14,30. You write down 1, the coefficient. You break off half of 1. 0;30 and 0;30 you multiply. You add 0;15 to 14,30. Result 14,30;15. This is the square of 29;30. You add 0;30, which you multiplied, to 29;30. Result: 30, the side of the square.

Again, this isn't a practical problem - we can't subtract the side of a square from the area.

Putting it into modern algebraic notation, we have x2 - x = 870, and carrying out the sequence of steps gives successively: 1, ½, (½)2 = ¼, 870¼, 29½, 30. It turns out that if we were to carry out the same operations on the general equation x2 - bx = c, we'd get the same result as we get nowadays from the quadratic equation formula. So the Mesopotamians knew how to find a solution of a specific quadratic equation 4000 years ago, using essentially the same method that we use today.

Another quadratic problem involves the igum and the igibum. The igibum exceeds the igum by 7. What are the igum and the igibum? This problem seeks a pair of reciprocals - or, equivalently, two numbers differing by 7 whose product is 60. In modern terms, we're asked to find two numbersx and y for which x - y = 7 and xy = 60. This quadratic problem can be solved much as before, and the answer is igibum = 12, igum = 5. It can also be solved geometrically, using the idea of completing the square.

The Greeks

Unlike the Egyptians and the Mesopotamians, the Greeks were more concerned with proving things, particularly in arithmetic and geometry, than with such algebraic pursuits as solving equations. But in Book II of his Elements, Euclid presents some propositions that are often described as 'geometrical algebra'.

One of these, Proposition 1, is the following: If there are two straight lines, and one of them is cut into any number of segments, the rectangle contained by the two lines is equal to the rectangles contained by the uncut straight line and each of the segments. If we write this in algebraic terms, with a as the height of the rectangle and b1, b2, ..., as the segments then the area is given by what we now call the distributive law:a (b1 + b2 + . . .) = ab1 + ab2 + ...

And Proposition 4 states: If a straight line is cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments. In our algebraic notation, with a + b as the side of the square, the total area (a + b)2 is the sum of the areas of the small squares a2 and b2 plus twice the area ab of the rectangle.

For another type of Greek algebra we turn to Diophantus of Alexandria. We don't know when he lived, possibly around 250 AD, but we do have a later puzzle from the Greek Anthology: Diophantus spent 1/6 of his life in childhood, 1/12 in youth, and 1/7 more as a bachelor. Five years after his marriage there was a son who died four years before his father, at ½ his father's final age. From this you can work out that he lived to be 84.

Diophantus wrote an important Arithmetic. Here's the title page of the 17th-century French translation by Bachet, which Fermat famously annotated in the margin, claiming to have a proof of what we now know as 'Fermat's last theorem'.

In his Arithmetic Diophantus solved a number of problems whose answers were to be given as whole numbers or fractions. For example, the equation 2x + 3y = 10 has infinitely many solutions, such as x = 55 and y= -331/3, but if we restrict ourselves to positive integers only, then there's just one solution: x = y = 2.

A typical example from the Arithmetic is: find two square numbers, the sum of which is a cubic number; the answer given is 25 + 100 = 125.

A much more complicated one is: find a right-angled triangle such that its perimeter is a square, while its perimeter added to its area gives a cube; the answer given is 1024/217, 215055/47089 and 309233/47089.

China

An important early text in China was the Nine Chapters on the Mathematical Art, an impressive work containing 246 questions with answers but with no working shown. It deals with practical matters (agriculture, business, surveying and engineering) and theoretical ones (areas and volumes of geometrical objects, calculating square and cube roots, the study of right-angled triangles, and solving simultaneous equations).

The treatment of this last topic is remarkable. Here's an example. There are three types of grain, good, moderate and poor. 3 bundles of good grain, 2 bundles of moderate grain, and 1 bundle of poor grain take up 39 measures; 2 bundles of good grain, 3 bundles of moderate grain, and 1 bundle of poor grain take up 34 measures; 1 bundle of good grain, 2 bundles of moderate grain, and 3 bundles of poor grain take up 26 measures. How many of each type are there?

These days we'd write down the three equations as done here, except that they're written from left to right and vertically. We'd then manipulate them, using what we now call 'Gaussian elimination' - and that's exactly what's done here, eventually yielding these three equations. The first gives C = 2¾, and substituting back gives B = 4¼ and A = 9¼. The Chinese method is exactly the same as Gauss's some 2000 years later, but Gauss is the one who got the credit!

Another Chinese topic was the Chinese remainder theorem, so-called because it apparently originated in China around 250 AD, when Sun Zi inMaster Sun's Mathematical Manual asked the following question. We have things of which we do not know the number. If we count them by 3s, the remainder is 2; if we count them by 5s, the remainder is 3; if we count them by 7s, the remainder is 2. How many things are there? Nowadays, we'd write the question in the form: find N, where N ≡ 2 (mod 3) ≡ 3 (mod 5) ≡ 2 (mod 7). Sun Zi gave the correct answer 23, and also explained how to solve the problem and others of the same type.

India

Moving on to India, around the 10th century Bhaskara asked: Tell me, O mathematician, what is that square which multiplied by 8 becomes -together with unity - a square. Here, 8x2 + 1 = y2, which has the easy solution x = 1, y = 3. This can then be used to find other solutions, such as x = 6 (so 8x2 + 1 = 289), giving y = 17.

More generally, Indian mathematicians looked at equations of the formCx2 + 1 = y2, where C is a given constant and x and y are integers; this is now called Pell's equation, even though John Pell had nothing to do with it. The hardest case they came across involved C = 67, but amazingly they still found solutions: the simplest is x = 5967 and y = 48,842.

Islamic mathematics

In Baghdad the caliphs promoted mathematics and astronomy - in particular, Caliph Harun al-Rashid and his son al-Ma'mun established the 'House of Wisdom', a scientific academy with its own extensive library and observatory.

One of its earliest scholars was Muhammad ben Musa al-Khwarizmi (c.780-850), the author of a number of influential treatises. He's remembered mainly for his books on arithmetic and algebra, which later had great influence in the West when translated into Latin.

The title of his algebra book is Kitab al-jabr wal-muqabala (the compendious book on calculation by completion and reduction). This is the origin of the word 'algebra': the Arabic word 'al-jabr' essentially refers to the operation of transposing a term from one side of an algebraic equation to the other. When this book was translated into Latin, its title became Ludus algebrae et almucgrabalaeque, so perhaps we got off lightly.

The Algebra commences with a lengthy account of linear and quadratic equations. Since negative numbers were still not considered meaningful, al-Khwarizmi split these equations into six types, corresponding (in modern notation) to the six forms ax2 = bx, ax2 = b, ax = b, ax2 + bx =c, ax2 + c = bx and ax2 = bx + c, where a, b and c are positiveconstants. He then solved particular instances of each type, such as x2 + 10x = 39, using a geometrical form of 'completing the square'.

The poet and mathematician Omar Khayyam (1048-1131) discussed equations in general, going from roots and squares to cubes, and then to square squares (quartic equations), square cubes, and so on. He systematically classified cubic equations and tried to solve one of the forma solid cube plus squares plus edges equal to a number (x3 + ax2 + bx =c) geometrically, by intersecting a conic with a hyperbola. But, cubic equations weren't solved in general for a further 500 years, as we'll see.

Europe

Over the centuries the Islamic world expanded along North Africa and up into Spain and Italy. In Pisa, Leonardo Fibonacci produced his Liber Abaci, designed to introduce the Hindu-Arabic numerals to his compatriots. As well as his famous 'rabbits problem', it contained a number of questions that we'd now solve algebraically, such as the following:

There is a tree, ¼ and 1/3 of which lie below ground; 21 palmi. How tall is the tree?

If I buy 3 sparrows for a penny, turtle-doves 2 for a penny, and doves for two pence, and spend 30 pence for 30 birds, how many of each kind do I buy?

If a lion eats a sheep in 4 hours, a leopard eats it in 5 hours, and a bear eats it in 6 hours, how long would they all take together?

Printed texts

With the invention of printing in the 15th century, algebra texts became much more widely available. Among these was Luca Pacioli's 1494Summa, a 600-page vernacular compilation of the mathematics known at the time, including algebra; it is also notable for containing the first published account of double-entry bookkeeping.

In this country, Robert Record was such a fine lecturer that his audience regularly applauded his lectures. His books were written in English and ran to many editions, and included his 1557 Whetstone of Witte on algebra:

Here if you lift your wittes to whette, Moche sharpenesse therby shall you gette.

Dulle wittes hereby doe greately mende, Sharpe wittes are fined to their fulle ende.

The appearance of popular books helped to establish terminology and notation, and it's here that we find the first appearance of our equals sign:And to avoide the tediouse repetition of these woordes: is equalle to: I will sette as I doe often in woorde use, a parre of paralleles, o: Gemowe lines of one lengthe, thus: == because noe 2 thynges can be moare equalle.

Cubic equations

Up to this time, there had been little progress in solving cubic equations, even though they arose in two of the ancient Greek classical problems -doubling the cube and trisecting the angle. Even around 1500, Pacioli and others were pessimistic about solving them in general.

We now come to one of the most celebrated stories in the history of our subject. The context is Italian mathematics in the early 16th century, at a time when university academics had no job security, frequently having to renew their positions on a yearly basis. To establish their credentials they took part in public

problem-solving contests in order to prove their superiority over other contenders - often, the winner would have to pay for thirty dinners for the loser and his friends - a sizeable sum.

In the 1520s Scipione del Ferro, a professor at the University of Bologna, found a general method for solving cubics of the form: a cube and things equal to numbers - that is, x3 + cx = d, where c and d are positive - and revealed his method to his pupil Antonio Fior.

After del Ferro's death in 1526, Fior felt free to exploit his secret, and challenged Niccolo of Brescia, known as Tartaglia ('the stammerer'), to a contest, presenting him with thirty cubics of this form and giving him a month to solve them. Tartaglia, who'd solved equations of the form cubes and squares equal to numbers (ax3 + bx2 = d), in turn presented Fior with thirty of these. Fior lost the contest: he was not a good enough mathematician to solve Tartaglia's type of problem, while Tartaglia, during a sleepless night ten days before the contest, discovered a method for solving all of Fior's problems. In order to help him to remember it, Tartaglia gave his method for x3 + cx = d in rhyme:

When the cube and the thing together

Are equal to some discrete number [x3 + cx = d]

Find two other numbers differing in this one. [u - v = d]

Then you will keep this as a habit

That their product shall always be equal

Exactly to the cube of a third of the things. [uv = (c/3)3]

The remainder then as a general rule

Of their cube roots subtracted

Will be equal to this principal thing. [x = u1/3 + v1/3]

We next meet Gerolamo Cardano, who wrote extensively about a range of topics, from medicine and the mathematics of gambling to arithmetic and algebra. Cardano heard about the contest, and was determined to get Tartaglia's method from him. He succeeded in doing so one evening in 1539 after promising to give him an introduction to Spanish Governor of the city: Tartaglia hoped that the Governor would fund his researches, and in turn extracted from Cardano a solemn oath not to reveal his methods:

I swear to you, by God's holy Gospels, and as a true man of honour, not only never to publish your discoveries, if you teach me them, but I also promise you, and I pledge my faith as a true Christian, to note them down in code, so that after my death, no-one will be able to understand them.

In the event, Cardano discovered in 1542 that Tartaglia's method was originally due to del Ferro, and felt free to break his oath. He published the method for solving cubics in his Ars Magna of 1545 - and also a method for quartics that had been found in the meantime.

The Ars Magna became one of the most important algebra books of all time, but Tartaglia was outraged and spent the remaining ten years of his life writing increasingly vitriolic letters and pamphlets to Cardano. Thus, after a struggle lasting many centuries, cubic equations had at last been solved, together with quartic equations.

One difficulty that arose in the solution of cubic equations was the appearance of the square roots of negative quantities. Cardano noticed these, saying 'Nevertheless we will operate with them, putting aside the mental tortures involved', and Rafael Bombelli discussed them in some detail in his algebra book, but they weren't fully understood for some centuries to come.

England and France

Around this time, the mathematical practitioner Thomas Harriot appeared on the scene, possibly the greatest English mathematician before Newton, with extensive writings on geometry and exciting new work on algebra - in particular, developing notation and solving equations.

Harriot has been called the founder of the English algebra school. In hisTreatise on equations he gave a systematic account of how to solve cubic and quartic equations, and was the first to recognise that all polynomials can be written as products of linear and quadratic factors, a topic to which I'll return next time.

This insight, combined with the clarity of his notation, enabled him to links between the roots and the coefficients of polynomial equations.

To Harriot we owe our symbols for < and >, the powers aa and aaa, and the cube-root sign. Almost all of his work is in manuscript, which is still being worked through. But although he published very little, his posthumous algebra book Artis analyticae praxis was very influential.

Algebraic developments were also taking place in France. François Viète pioneered an improvement in notation, using letters for unknowns, rather than writing them in words. He insisted on the importance of dimension, dismissing earlier problems that involved 'adding lines to areas' (meaningax2 + bx). His use of letters was later extended by Descartes, who wrote powers in the modern way (x3, x4, etc.), although he preferred xx to x2. Descartes also investigated geometrical ways of representing square roots and roots of quadratic equations.

Descartes also produced his 'rule of signs', which was later developed by Newton. As an example of this, consider the quartic equation x4 - 4x3 - 19x2 + 106x - 120 = 0. Moving from left to right, we find three sign-changes and one non-change of sign. From this we deduce that there are up to three positive solutions and one negative one. This is correct: the solutions are 2, 3, 4 and -5.

Another French mathematician of the time was Pierre de Fermat, who made spectacular advances in number theory and analytic geometry, although he didn't publish his results. In number theory he's remembered for his 'little theorem', that for any integer a and prime number p, ap - ais always divisible by p: for example, 1437 - 14 is divisible by 37. He's also remembered for his famous claim to have proved what is now known as 'Fermat's last theorem': for any value of n > 2, there are no positive integers x, y and z that satisfy the equation xn + yn = zn. He indeed proved this result in the special case n = 4, but a full proof had to wait until 1995, when it was obtained by Andrew Wiles.

Abel and Galois

There were also other important developments in France concerning the solution of equations. We've already discussed quadratic, cubic and quartic equations, and shown that there are general methods that can be used to solve them involving only arithmetical operations and extracting roots.

But how about equations of degree 5 or more? The search for a general solution or formula for these had occupied the finest mathematicians, such as Descartes and Euler, but little progress was made until Lagrange attempted the problem, laying the groundwork for the eventual solution.

Lagrange's first approach was as follows. Given a cubic equation such asx3 + nx + p = 0, we can substitute x = y - n/(3y) and the equation becomes y6 + py3 - 1/27n3 = 0, a quadratic equation in y3 that we can solve; the six values for y give six values of x, but these turn out to be equal in pairs, so we get just three values for x, as expected.

Similarly, given a quartic equation x4 + nx2 + px + q = 0, we can reduce it to the equation

y3 - 1/2n - qy + 1/8(4nq - p2) = 0, a cubic equation that we can solve; we eventually obtain the required values for x.

Lagrange found that this approach does not work for the quintic equation of degree 5, so he took another approach. Suppose that we are given the cubic equation x3 + ax2 + bx + c = 0, and that its solutions are p, q andr, so that

x3 + ax2 + bx + c = (x - p) (x - q) (x - r).

If we multiply out the right-hand side, we find that the constant term c is -pqr, the x-term b is pq + pr + qr, and the x2-term a is -(p + q + r). Each of these is symmetrical in p, q and r, which means that if we permute p, qand r in any way we like, we get just one possible value in each case. But if we take the expression pq + 2r, we don't get different values when we permute p and q and leave r - in fact, we can get just three different values: pq + 2r, pr + 2q and qr + 2p. Or if we take p + 2q + 5r, we get six different values.

So the number of possible different values can vary, but it always divides 6 - and in general, if we have an equation of the nth degree, rather than a cubic, then the number of different values we can obtain always dividesn!. This result later became known in a more general setting asLagrange's theorem for groups.

This idea of permuting the solutions and counting the number of different values was taken up by Paolo Ruffini, who published the bold claim that:The algebraic solution of equations of degree greater than 4 is always impossible. Behold a very important theorem which I believe I am able to assert... The immortal Lagrange, with his sublime reflections, has provided the basis of my proof. Unfortunately, Ruffini's proof contained a gap, and it was left to others to continue the work.

It was not until the 1820s that a proof finally emerged, that there is nogeneral method, or formula, for solving equations of degree 5 or more. This was obtained by the Norwegian mathematician Niels Henrik Abel, developing Lagrange's ideas.

Abel had a short and tragic life. Growing up in Norway, he was desperate to study in the centres of mathematical life, France and Germany. He was eventually able to spend a little time in both countries, and contributed several works to a new mathematical journal founded in Germany by his friend Crelle; among these was his proof of the impossibility for solving the quintic equation. He returned to Norway where he contracted tuberculosis and died at the age of only 26. Two days later a letter arrived at his home offering him a professorship at Berlin.

Abel's work was continued by the brilliant young French mathematician Évariste Galois, who obtained criteria for deciding which quintic and other equations can be solved. These ultimately led to whole new areas of algebra, now known as group theory and Galois theory.

Galois also had a short and tragic life, dying in a duel of honour at the age of 20. A firebrand who became involved following the July Revolution of 1830, he threatened the life of the king, Louis-Phillippe, but was acquitted. A month later he appeared on Bastille Day in the uniform of the banned Artillery guard and carrying several weapons, whereupon he was thrown into jail. The night before his duel, arising from his supposed involvement with a young lady, he frantically wrote out his mathematical achievements for posterity, but it was some years before anyone appreciated what they meant and what a genius the world had lost.

Victorian algebra

I'd now like to mention a couple of results from Victorian Britain. The topic of matrices was developing around this time, arising from the concept of a determinant. Determinants arose in the 17th century in Japan with Takakazu Seki and later in Europe with Leibniz.

Determinants arise in the solution of simultaneous equations. Given four numbers a, b, c, d, we can calculate their determinant ad - bc. Next, given two equations such as $2x + y = 5$ and $x + 2y = 4$, we can calculate the determinants of 2, 1, 1, 2 (= 3), 5, 4, 1, 2 (= 6) and 2, 1, 5, 4 (= 3). The solutions are then given by $x = 6/3 = 2$ and $y = 3/3 = 1$. We can similarly calculate the determinants that arise from larger collections of simultaneous equations; Charles Dodgson (Lewis Carroll), in particular, invented a method for doing so.

Around this time, the English mathematician Arthur Cayley studied the algebraic properties of matrices - rectangular arrays of numbers used to represent transformations of the plane and of space - and in 1858 wrote a pioneering paper on the algebra of matrices.

The mathematics of symmetry

The ideas of Lagrange, Cauchy, Galois and others led to the area of algebra known as group theory, which has been called the mathematics of symmetry. To introduce the idea, let's look at the English art of bell-ringing.

The method known to bell-ringers as Plain Bob Minimus involves the ringing of four bells (which we'll call 1, 2, 3, 4) in such a way as to include all twenty-four possible permutations of these four numbers. Because bells are heavy, We can interchange only bells that are consecutive, so if we start with the bells in the order 1234, we could interchange both outer pairs of bells 12 and 34 (giving 2143) or the inner pair 23 (giving 1324). Thus, we could start

$1234 \rightarrow 2143 \rightarrow 2413 \rightarrow 4231 \rightarrow 4321 \rightarrow 3412 \rightarrow 3142 \rightarrow 1324.$

If we now interchanged the middle bells we'd get back to the starting point, so instead we interchange just the last two, giving 1342, and then proceed as before with 3124, 3214, and so on. Eventually we obtain all

24 permutations of 1, 2, 3, 4.

More generally, we can look at all the permutations on 1, 2, 3, 4, 5 (there are 120 of these), or of any collection of symbols. Some permutations are even, in that they are obtained by switching an even number of pairs (such as the pairs 12 and 34), while others are odd (such as switching the single pair 23). We combine two permutations by carrying out one after the other, or we can form the inverse of a permutation by undoing it (for example, switching the bells back).

A different type of example gives rise to similar conclusions. Suppose we look at all the rotations of a cube lying on a table. There are 24 of these, since each of the six faces can be in contact with the table and can be in any of four different positions: these 24 rotations consist of the 'identity' (where we leave the cube as it is), 9 face-rotations (three rotations about each of the three opposite pairs of faces), 8 vertex-rotations (two rotations about a line joining each of the four opposite pairs of corners) and 6 edge-rotations (one about a line joining the midpoints of each of the six opposite pairs of edges). In fact, if we consider the four diagonals of the cube, we can check that each permutation of these diagonals corresponds exactly to one of the bell-ringing permutations we had earlier.

Similarly, we can look at the rotations of other geometrical figures, such as a dodecahedron (with twelve pentagonal faces); this gives 60 rotations.

What do all these examples have in common? In each case we have a set of objects (such as permutations, or rotations) which we can combine in pairs to give another object of the same type, and they satisfy three further properties: there's an identity element (where we 'do nothing'), each rotation or permutation has an inverse (it can be 'undone'), and combinations are associative (if we do the first two followed by the third, it's the same as doing the first followed by the second and third). These give rise to the following axioms for an abstract group:

Take a set G of elements, and a rule (denoted by ?) for combining them. For a group:

Closure: if a and b are in G, then so is a ? b.

Associativity: if a, b and c are in G, then (a ? b) ? c = a ? (b ? c).

Identity: there is an element e such that a ? e = e ? a = a, for all a in G.

Inverses: for each a in G, there is an inverse a-1 such that a ? a-1 = a-1? a = e.

For many groups (but not these), we get the same result when we combine elements in either order:

Abelian-ness (or Commutativity): if a and b are in G, then a ? b = b ? a.

An example of an Abelian group is the addition of integers ..., -2, -1, 0, 1, 2, 3, ... ; the identity element is 0 and the inverse of a is -a. As an example of associativity we see that (3 + 4) + 5 = 3 + (4 + 5), and the group is Abelian - for example, 3 + 4 = 4 + 3.

We next consider the rotation of regular polygons. If we rotate a square through 0, 1, 2 or 3 right angles, it ends up in the same position. We can 'add' these rotations: 2 right angles + 3 right angles = 1 right angle, and so on; this is the same as adding the integers 0, 1, 2, 3 ?modulo 4? ? that is, we add, and then remove multiples of 4 to get back to 0, 1, 2 or 3, so 3 + 3 = 2, 1 + 3 = 0, etc. This gives a group called the cyclic group C4. Similarly, we can rotate a regular pentagon through multiples of 72° to get the cyclic group C5, consisting of the integers 0, 1, 2, 3, 4, which we add modulo 5. More generally, starting with a regular n-sided polygon we get the cyclic group Cn.

In the 20th century, algebraists were concerned with classifying groups of various types - such as Abelian groups. Amazingly, we can obtain anyAbelian group by taking cyclic groups and combining them in a simple way: this is called the classification theorem for Abelian groups.

The ultimate goal is to classify all groups by showing how they can be constructed from basic 'building blocks' - just as all natural numbers can be formed by multiplying prime numbers, and all Abelian groups can be formed by combining cyclic groups. The basic building blocks for groups in general are called simple groups - these are groups that essentially contain no smaller groups inside them (technically, they have no non-trivial 'normal subgroups' - a concept introduced around 1830 by Galois).

But which groups are simple? Much of the 20th century was spent in trying to classify all the simple groups, and this quest was ultimately successful - a major achievement involving thousands of pages of

detailed work!

To cut a long story short, there are several infinite classes of simple groups - such as all the cyclic groups with a prime number of elements (for example, C5, C37, C101, ...), or the rotations of a dodecahedron, or the set of all even permutations of a set of five or more elements, or various groups of 'Lie type' arising from matrices - together with 26 individual so-called 'sporadic groups' that don't fit into any of these infinite categories. The largest of these sporadic groups is called 'the Monster group', which has:

808,017,424,794,512,875,886,459,904,061,710,757,005,754,368,000,000,000 elements.