INTRODUCTION

I have discovered things so wonderful that I was astounded ... Out of nothing I have created a strange new world.

János Bolyai

With these words the young Hungarian mathematical prodigy János Bolyai, reputedly the best swordsman and dancer in the Austrian Imperial Army, wrote home about his discovery of non-Euclidean geometry in 1823. Bolyai's discovery indeed marked a turning point in history, and as the century progressed mathematics finally freed itself from the lingering sense that it must describe only the patterns in the 'real' world. Some of the doors which these discoveries flung open led directly to new worlds whose full exploration has only become possible with the advent of high speed computing in the last twenty years.

Paralleling the industrial revolution, mathematics grew explosively in the nineteenth century. As yet, there was no real separation between pure and applied mathematics. One of the main themes was the discovery and exploration of the many special functions (sines, cosines, Bessel functions and so on) with which one could describe physical phenomena like waves, heat and electricity. Not only were these functions useful, but viewed as more abstract entities they took on a life of their own, displaying patterns whose study intrigued many people. Much of this had to do with understanding what happened when ordinary 'real' numbers were replaced by 'complex' ones, to be described in Chapter 2.

A second major theme was the study of symmetry. From Mayan friezes to Celtic knotwork, repeating figures making symmetrical patterns are as ancient as civilization itself. The Taj Mahal reflects in its pool, floors are tiled with hexagons. Symmetry abounds in nature: butterfly wings make perfect reflections and we describe the tile pattern as a honeycomb. The ancients already understood the geometry of symmetry: Euclid tells us how to recognise by measurement when two triangles are congruent or 'the same' and the Alhambra displays all mathematically possible ways of covering a wall with repeating tiles.

The nineteenth century saw huge extensions of the idea of symmetry and congruence, drawing analogies between the familiar Euclidean world and others like Bolyai's new non-Euclidean universe.[1] Around the middle of the century, the German mathematician and astronomer August Möbius had the idea that things did not have to be the same shape to identified: they could be compared as long as there was a definite *verwandschaft* or 'relationship' between every part of one figure and every part of the other. One particular new relationship studied by Möbius was inspired by cartography: figures could be considered 'the same' if they only differed by the kind of distortions you have to make to project figures from the round earth to the flat plane. As Möbius pointed out (and as we shall study in Chapter 3), these special relationships, now called Möbius maps, could be manipulated using simple arithmetic with complex numbers. His constructs made beautifully visible the geometry of the complex plane.[2]

[1] Non-Euclidean geometry was actually discovered independently and at more or less the same time by Gauss, Bolyai and Lobachevsky, see Chapter 12.

[2] The complex plane is pictured in Figure 2.1

Towards the end of the century, Felix Klein, one of the great mathematicians his age and the hero of our book, presented in a famous lecture at Erlangen University a unified conception of geometry which incorporated both Bolyai's brave new world and Möbius' relationships into a wider conception of symmetry than had ever been formulated before. Further work showed that his symmetries could be used to understand many of the special functions which had proved so powerful in unravelling the physical properties of the world (see Chapter 12 for an example). He was led to the discovery of symmetrical patterns in which more and more distortions cause shrinking so rapid that an infinite number of tiles can be fitted into an enclosed finite area, clustering together as they shrink down to infinite depth.

It was a remarkable synthesis, in which ideas from the most diverse areas of mathematics revealed startling connections. Moreover the work had other ramifications which were not to be understood for almost another century. Klein's books (written with his former student Robert Fricke) contain many beautiful illustrations, all laboriously calculated and drafted by hand. These pictures set the highest standard, occasionally still illustrating mathematical articles even today. However many of the objects they imagined were so intricate that Klein could only say:

> The question is ... what will be the position of the limiting points. There is no difficulty in answering these questions by purely logical reasoning; but the imagination seems to fail utterly when we try to form a mental image of the result.[3]

[3] *The mathematical character of space-intuition*, Klein, *Lectures on Mathematics*, 1894, Reprinted by AMS Chelsea, 2000.

The wider ramifications of Klein's ideas did not become apparent until two vital new and intimately linked developments occurred in the 1970's. The first was the growing power and accessibility of high speed computers and computer graphics. The second was the dawning realization that chaotic phenomena, observed previously in isolated situations (such as theories of planetary motion and some electronic circuits), were ubiquitous, and moreover provided better models for many physical phenomena than the classical special functions. Now one of the hallmarks of chaotic phenomena is that structures which are seen in the large repeat themselves indefinitely on smaller and smaller scales. This is called self-similarity. Many schools of mathematics came together in working out this new vision but, arguably, the computer was the *sine qua non* of the advance, making possible as it did computations on a previously inconceivable scale. For those who knew Klein's theory, the possibility of using modern computer graphics to actually *see* his 'utterly unimaginable' tilings was irresistible.

Our frontispiece is a modern rendering of one of Klein's new symmetrical worlds. In another guise, it becomes the *The Glowing Limit* shown overleaf. Peering within the bubbles, you can see circles within circles, evoking an elusive sense of symmetry alongside the self-similarity characteristic of chaos. Without the right mathematical language, though, it is hard to put one's finger on exactly what this symmetry is. The sizes and positions of the circles in the two pictures are not the same: the precise *verwandschaft* between them results from the distortion allowed by a Möbius map.

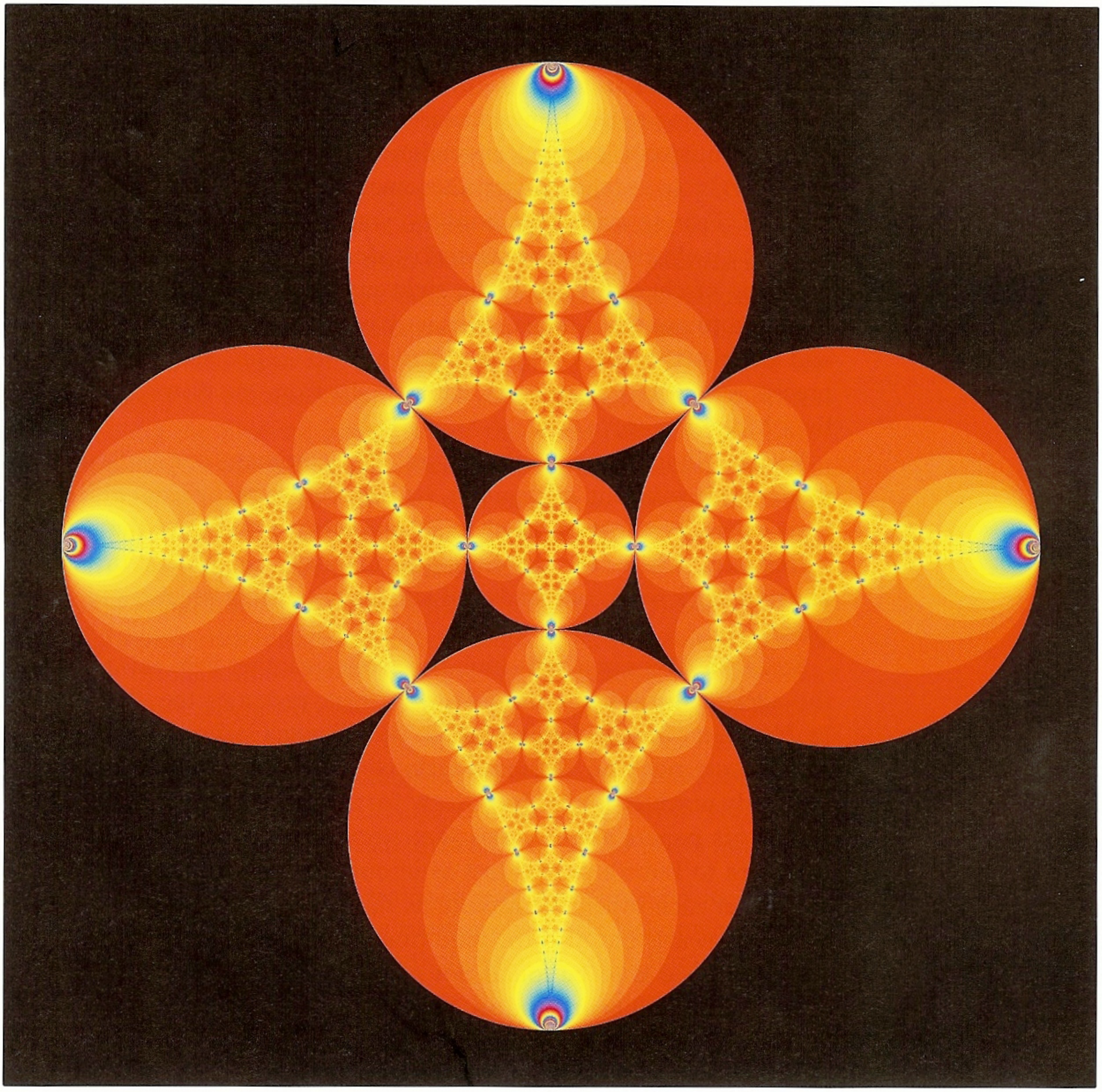
Klein's tilings were now seen to have intimate connections with modern ideas about self-similar scaling behaviour, ideas which had their origin in statistical mechanics, phase transitions and the study of turbulence. There, the self-similarity involved random perturbations, but in Klein's work, one finds self-similarity obeying precise and simple laws.

Strangely, this *exact* self-similarity evokes another link, this time with the ancient metaphor of Indra's net which pervades the *Avatamsaka* or *Hua-yen Sutra*, called in English the *Flower Garland Scripture*, one of the most rich and elaborate texts of East Asian Buddhism. We are indirectly indebted to Michael Berry for making this connection: it was in one his papers about chaos that we first found the reference from the Sutra to Indra's pearls. Just as in our frontispiece, the pearls in the net reflect each other, the reflections themselves containing not merely the other pearls but also the reflections of the other pearls. In fact the entire universe is to be found not only in each pearl, but also in each reflection in each pearl, and so *ad infinitum*.

As we investigated further, we found that Klein's entire mathematical set up of the same structures being repeated infinitely within each other at ever diminishing scales finds a remarkable parallel in the philosophy and imagery of the Sutra. As F. Cook says in his book *Hua-yen: The Jewel Net of Indra*:

> The Hua-yen school has been fond of this mirage, mentioned many times in its literature, because it symbolises a cosmos in which there is an infinitely repeated interrelationship among all the members of the cosmos. This relationship is said to be one of simultaneous *mutual identity* and *mutual intercausality*.

The Glowing Limit. This illustration follows the mantra of Indra's Pearls *ad infinitum* (at least in so far as a computer will allow). The glowing yellow lacework manifests entirely of its own accord out of our initial arrangement of just five touching red circles.



In the words of Sir Charles Eliot:

> In the same way each object in the world is not merely itself but involves every other object and in fact *is* everything else.

Making a statement equally faithful to both mathematics and religion, we can say that each part of our pictures contains within itself the essence of the whole.

Perhaps we have been carried away with this analogy in our picture *The Glowing Limit* in which the colours have been chosen so that the cluster points of the minutest tiles light up with a mysterious glow. Making manifest of the philosophy of the Sutra, zooming in to any depth (as you will be able to do given your own system to make the programs), you will see the same lace-like structure repeating at finer and finer levels, worlds within worlds within worlds. The glowing pattern is a 'fractal', called the 'limit set' of one of Klein's symmetrical iterative procedures. How to understand and draw such limit sets is what this book is all about.

Like many mathematicians, we have frequently felt frustration at the difficulty of conveying the excitement, challenge and creativity involved in what we do. We hope that this book may in some small way help to redress this balance. On whatever level you choose to read it, be it leafing through for the pictures, reading it casually, playing with the algebra, or erecting a computer laboratory of your own, we shall have succeeded if we have conveyed something of the beauty and fascination of exploring this God-given yet man-made universe which is mathematics.