4000 YEARS OF ALGEBRA

Robin Wilson

The Open University

x C N q

$x + \frac{1}{2}x = 16$

y^2 Z $\sqrt{d}$ a^3

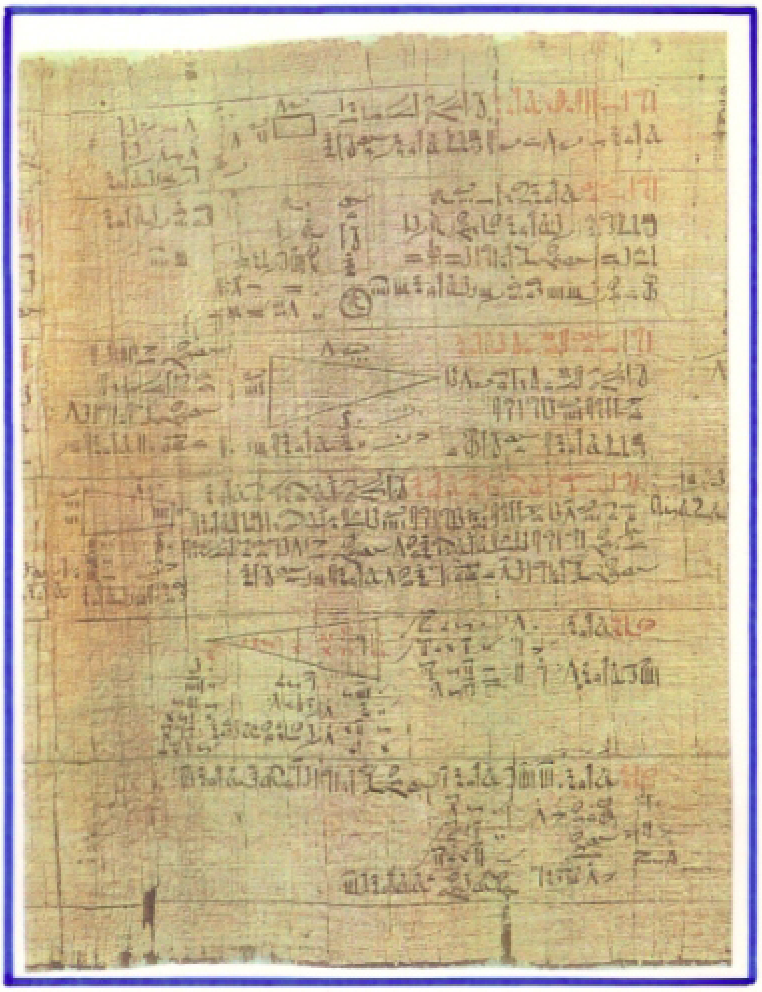
$e \cdot x = x$

b $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ x y

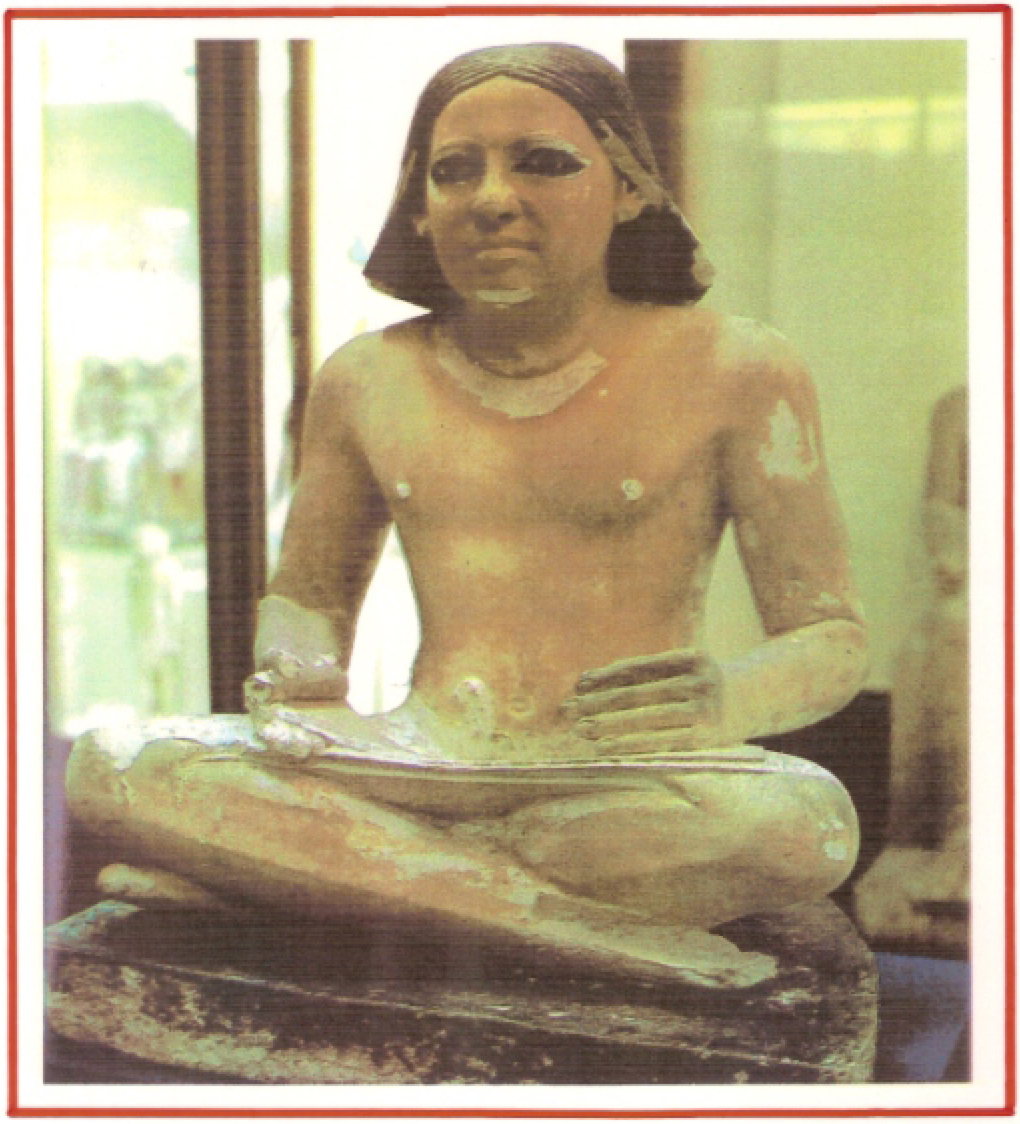
$\sqrt{b^2 - 4ac}$

Rhind Papyrus

(c. 1650 BC)



Egyptian Scribe



Problem 25

A quantity and its $\frac{1}{2}$ added together become 16.

What is the quantity? $x + \frac{1}{2}x = 16$

Assume 2

\	1	2
\	$\frac{1}{2}$	1
	Total	3

method of false position

As many times as 3 must be multiplied to give 16, so many times 2 must be multiplied to give the required number.

\	1	3	
	2	6	
\	4	12	16
	$\frac{2}{3}$	2	
\	$\frac{1}{3}$	1	
	Total $5\frac{1}{3}$		
	1	$5\frac{1}{3}$	
\	2	$10\frac{2}{3}$	

Do it thus:

The quantity is	$10\frac{2}{3}$
$\frac{1}{2}$	$5\frac{1}{3}$
Total	16

Egyptian fractions

Unit fractions:

(reciprocals)

$\frac{1}{n}$ (and $\frac{2}{3}$)

$$\frac{2}{11} = \frac{1}{6}\ \frac{1}{66}$$

$$\frac{2}{13} = \frac{1}{8}\ \frac{1}{52}\ \frac{1}{104}$$

Rhind papyrus, Problem 31

A quantity, its $\frac{2}{3}$, its $\frac{1}{2}$, and its $\frac{1}{7}$, added together, become 33.

What is the quantity?

[Solve: $x + \frac{2}{3}x + \frac{1}{2}x + \frac{1}{7}x = 33$]

Solution: The total is

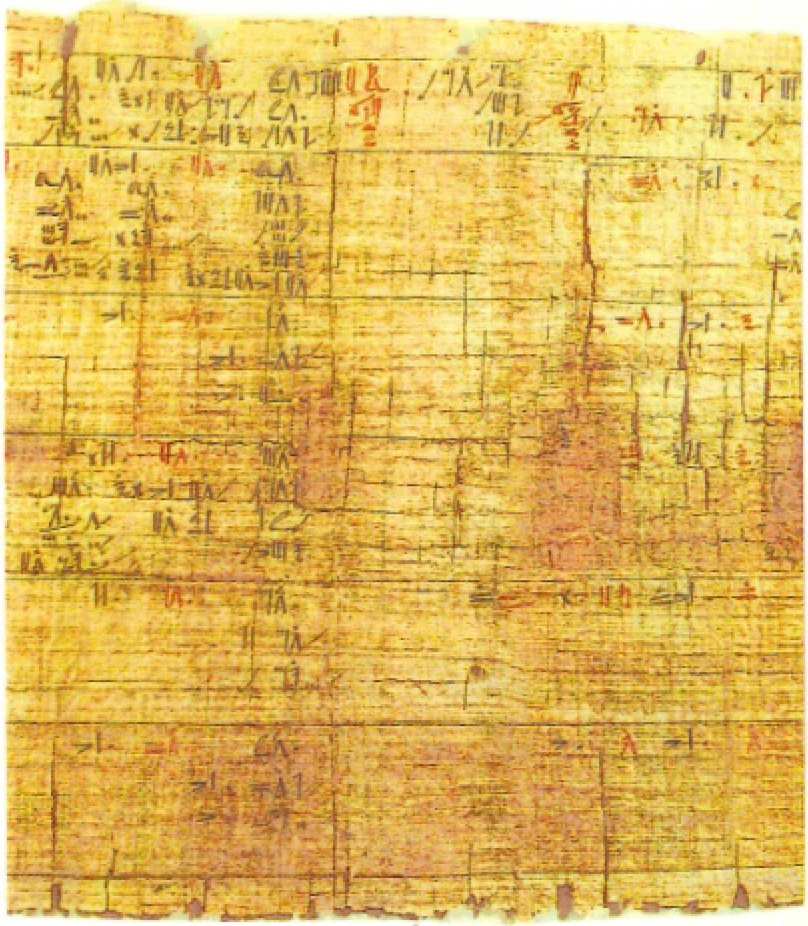
$$14\ \frac{1}{4}\ \frac{1}{56}\ \frac{1}{97}\ \frac{1}{194}\ \frac{1}{388}\ \frac{1}{679}\ \frac{1}{776},$$

$14\frac{28}{97}$

which multiplied by $1\ \frac{2}{3}\ \frac{1}{2}\ \frac{1}{7}$ makes 33.

Table of Fractions

$^2/n$, for $n = 5, 7, 9, 11, \cdots, 99, 101$



Mesopotamian Mathematics

clay tablets – cuneiform writing

place-value system based on 60 : < , |

<<<<|<<<< $= 41(60)+40$, or $41\frac{40}{60}$, or ...

Weighing a Stone

I found a stone, but did not weigh it;

after I weighed out 6 times its weight,

added 2 gin,

and added one-third of one-seventh

multiplied by 24,

I weighed it: 1 ma-na.

What was the original weight of the stone?

[Tablet had 22 such problems: 1 ma-na = 60 gin]

Solution:

$$(6x+2) + \frac{1}{3}\cdot\frac{1}{7}\cdot 24\,(6x+2) = 60 \text{ gin}$$

so $x = 4\frac{1}{3}$ gin.

Check: $28 + \frac{1}{3}\cdot\frac{1}{7}\cdot 24\cdot 28 = 28+32 = 60.$

Solving a 'Quadratic Equation'

I have subtracted the side of my square from the area: 14,30.

You write down 1, the coefficient.

You break off half of 1. 0;30 and 0;30 you multiply. You add 0;15 to 14,30.

Result 14,30;15. This is the square of 29;30.

You add 0;30, which you multiplied, to 29;30.

Result: 30, the side of the square.

$x^2 - x = 870$:

$$1 \rightarrow \tfrac{1}{2} \rightarrow \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4} \rightarrow 870\tfrac{1}{4} \rightarrow 29\tfrac{1}{2} \rightarrow 30.$$

$x^2 - bx = c$:

$$b \rightarrow \frac{b}{2} \rightarrow \left(\frac{b}{2}\right)^2 \rightarrow \left(\frac{b}{2}\right)^2 + c \rightarrow \sqrt{\left(\frac{b}{2}\right)^2 + c}$$

$$\rightarrow \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + c}.$$

The igum and the igibum

The igibum exceeds the igum by 7.

What are the igum and the igibum?

$\boxed{x-y=7}$ $\boxed{xy=60}$

Halve 7, and the result is 3;30.

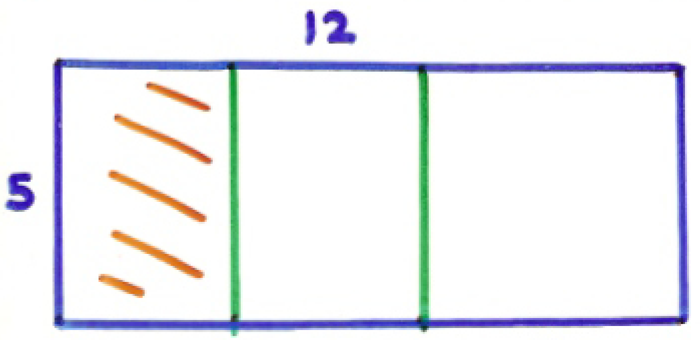
Multiply 3;30 with 3;30, and we get 12;15.

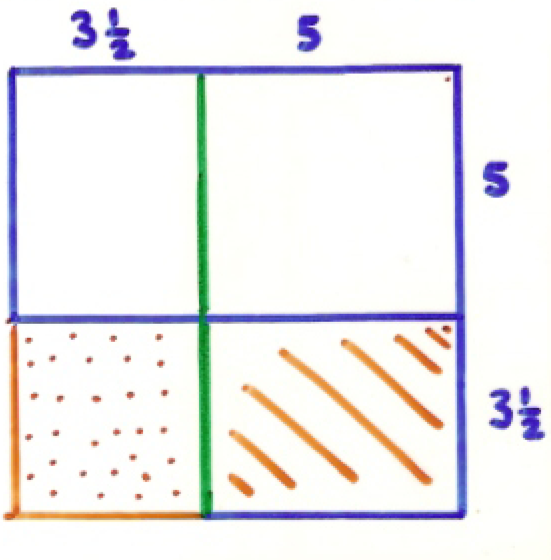
Add 1,0, the product, and we get 1,12;15.

What is the square root of 1,12,15? 8;30

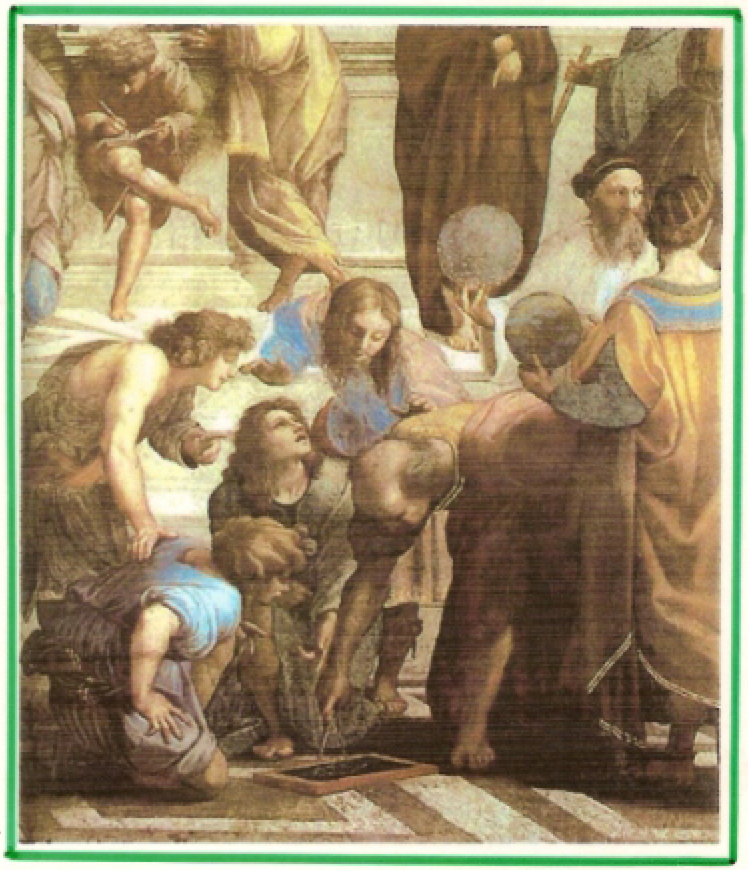
Lay down 8;30 and 8;30 and subtract 3;30 from one and add it to the other.

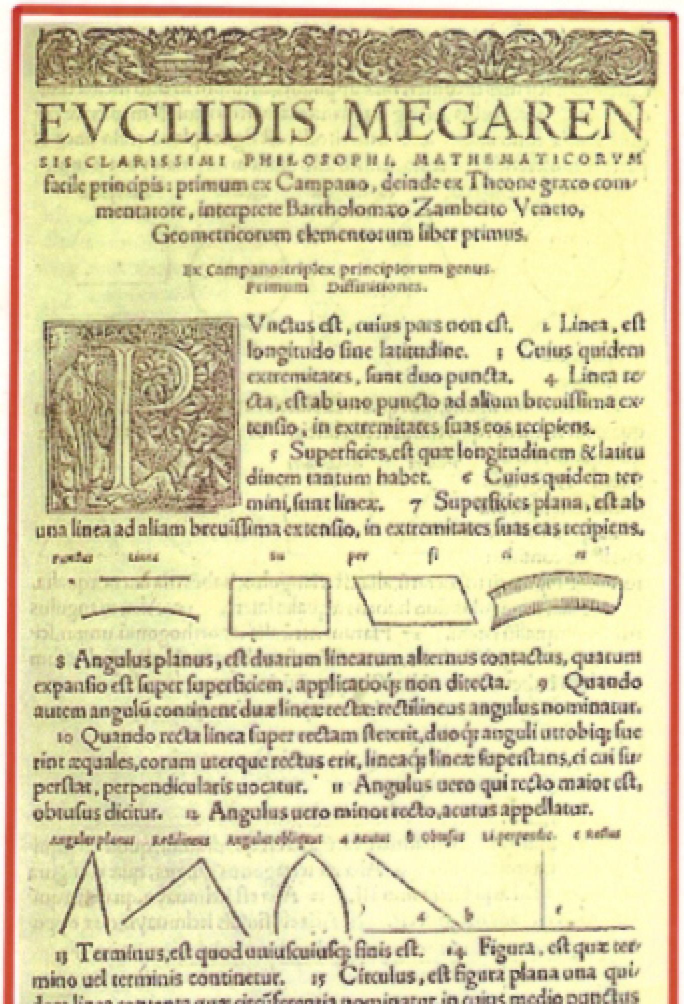
One is 12 (the igibum), the other 5 (the igum).





$$x, y = \sqrt{(3\tfrac{1}{2})^2 + 60} \pm 3\tfrac{1}{2} = 12 \text{ or } 5.$$





EVCLIDIS MEGAREN
SIS CLARISSIMI PHILOSOPHI, MATHEMATICORVM
facile principis: primum ex Campano, deinde ex Theone græco commentatore, interprete Bartholomæo Zamberto Veneto,
Geometricorum elementorum liber primus.

Ex Campano triplex principiorum genus.
Primum Diffinitiones.

PVnctus est, cuius pars non est. 2 Linea, est longitudo sine latitudine. 3 Cuius quidem extremitates, sunt duo puncta. 4 Linea recta, est ab uno puncto ad alium breuissima extensio, in extremitates suas eos recipiens.
5 Superficies, est quæ longitudinem & latitudinem tantum habet. 6 Cuius quidem termini, sunt lineæ. 7 Superficies plana, est ab una linea ad aliam breuissima extensio, in extremitates suas eas recipiens.

8 Angulus planus, est duarum linearum alternus contactus, quarum expansio est super superficiem, applicatioq; non directa. 9 Quando autem angulū continent duæ lineæ rectæ: rectilineus angulus nominatur.
10 Quando recta linea super rectam steterit, duoq; anguli utrobiq; fuerint æquales, eorum uterque rectus erit, lineaq; lineæ superstans, ei cui superstat, perpendicularis uocatur. 11 Angulus uero qui recto maior est, obtusus dicitur. 12 Angulus uero minor recto, acutus appellatur.

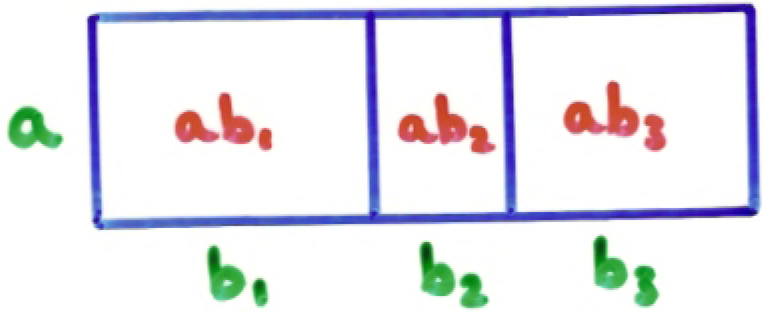
13 Terminus, est quod uniuscuiusq; finis est. 14 Figura, est quæ termino uel terminis continetur. 15 Circulus, est figura plana una quidem linea contenta quæ circūferentia nominatur, in cuius medio punctus est, à quo omnes lineæ rectæ ad circumferentiam exeuntes, sibi inuicem

Euclid

(c. 300 BC)

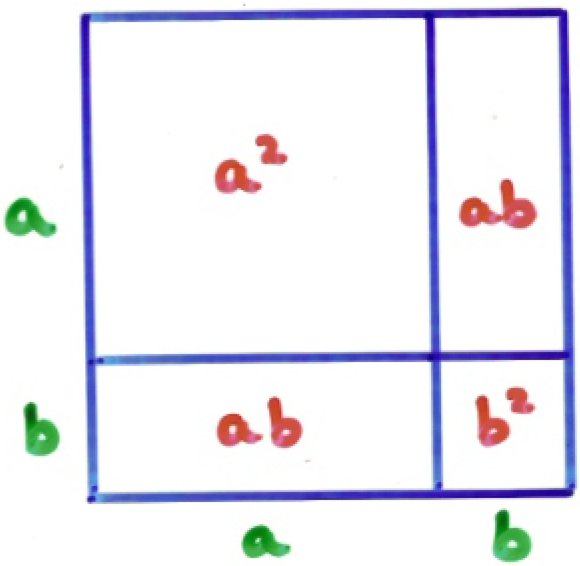
Euclid : Book II

II.1



$$a(b_1 + b_2 + b_3) = ab_1 + ab_2 + ab_3$$

II.4



$$(a+b)^2 = a^2 + b^2 + 2ab$$

How old was Diophantus?

Diophantus spent $1/6$ of his life in childhood, $1/12$ in youth, and $1/7$ more as a bachelor.

Five years after his marriage there was a son who died four years before his father at $1/2$ his father's final age.

x = Diophantus's age:

$$\left(\frac{1}{6}x + \frac{1}{12}x + \frac{1}{7}x\right) + 5 + \frac{1}{2}x + 4 = x,$$

so $\underline{x = 84 \text{ years}}$

DIOPHANTI ALEXANDRINI ARITHMETICORVM LIBRI SEX,

ET DE NVMERIS MVLTANGVLIS LIBER VNVS.

CVM COMMENTARIIS C. G. BACHETI V. C. & obſeruationibus D. P. de FERMAT *Senatoris Toloſani.*

Acceſſit Doctrinæ Analyticæ inuentum nouum, collectum ex varijs eiuſdem D. de FERMAT Epiſtolis.

TOLOSÆ,
Excudebat BERNARDVS BOSC, è Regione Collegij Societatis Ieſu.

M. DC. LXX.

Problems of Diophantus

Solutions in whole numbers or fractions:

$2x + 3y = 10$: $x = 2,\ y = 2$

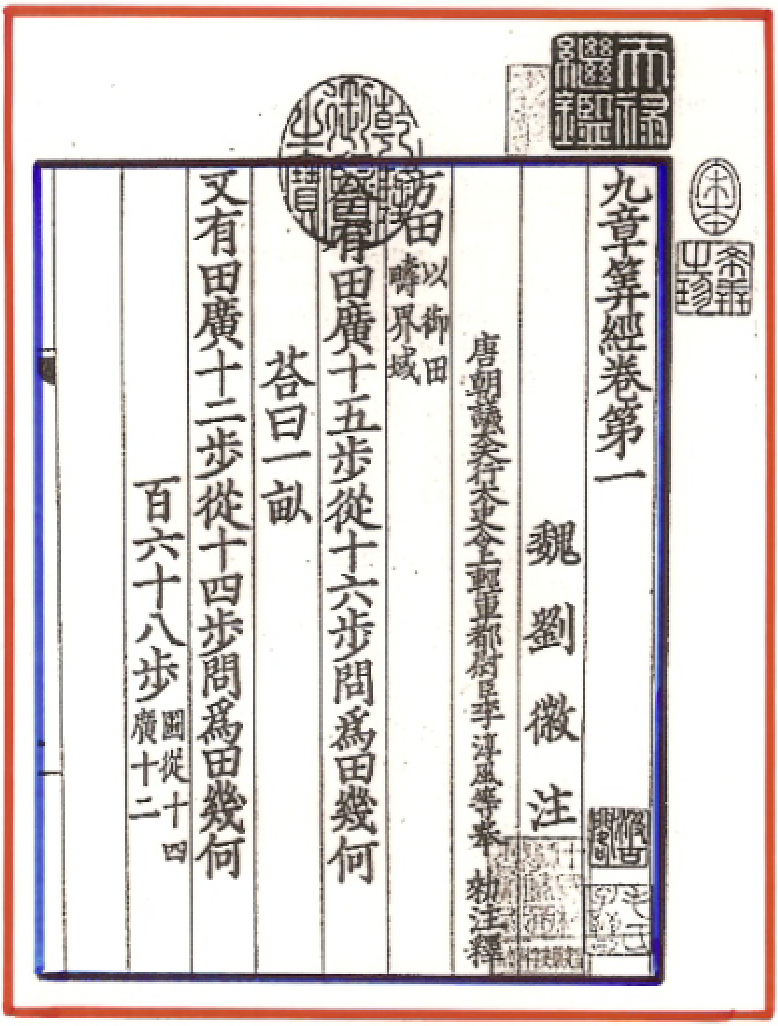
Find two square numbers, the sum of which is a cubic number:

$$25 + 100 = 125$$

Find a right-angled triangle such that its perimeter is a square, while its perimeter added to its area gives a cube.

$$\frac{1024}{217},\ \frac{215055}{47089},\ \frac{309233}{47089}$$

Jiuzhang suanshu



九章筭經卷第一

魏 劉徽 注

唐朝議大夫行太史令上輕車都尉臣李淳風等奉勅注釋

方田 以御田疇界域

有田廣十五步從十六步問爲田幾何

荅曰一畝

又有田廣十二步從十四步問爲田幾何

百六十八步 圖從十四 廣十二

Opening of Chapter 1

Nine Chapters on the Mathematical Art

(200 BC?)

246 questions with answers (but no 'working')

agriculture, business, surveying, etc

- calculation of areas and volumes
- calculation of square/cube roots
- study of right-angled triangles
- simultaneous equations

A problem involving grain

Three types of grain: A, B and C.

3 bundles of A, 2 of B, 1 of C = 39 measures;

2 bundles of A, 3 of B, 1 of C = 34 measures;

1 bundle of A, 2 of B, 3 of C = 26 measures.

How many measures in one bundle of each type?

(1) $3A + 2B + 1C = 39$

(2) $2A + 3B + 1C = 34$

(3) $1A + 2B + 3C = 26$

𝍠	𝍡	𝍢	(A)
𝍡	𝍢	𝍡	(B)
𝍢	𝍠	𝍠	(C)
𝍪𝍥	𝍫𝍣	𝍫𝍨	
(3)	(2)	(1)	

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ 26 & 34 & 39 \end{matrix}$$

$$\downarrow$$

$$\begin{matrix} 0 & 0 & 3 \\ 0 & 5 & 2 \\ 36 & 1 & 1 \\ 99 & 24 & 39 \end{matrix} \rightarrow \begin{matrix} 36C = 99 \\ 5B + C = 24 \\ 3A + 2B + C = 39 \end{matrix} \rightarrow \left\{ \begin{matrix} C = 2\,3/4 \\ B = 4\,1/4 \\ A = 9\,1/4 \end{matrix} \right.$$

The Chinese Remainder Theorem

Sun Zi (250 AD) in Sunzi suanjing

(Master Sun's mathematical manual).

We have things of which we do not know the number.

If we count them by 3s, the remainder is 2.

If we count them by 5s, the remainder is 3.

If we count them by 7s, the remainder is 2.

How many things are there?

$$N \equiv 2 \pmod{3} \equiv 3 \pmod{5} \equiv 2 \pmod{7}$$

The answer is 23.

Bhaskara (1100AD) : 'Pell's equation'

Tell me, O mathematician, what is that square which multiplied by 8 becomes – together with unity – a square?

$$8x^2 + 1 = y^2$$

Solutions : $x=1, y=3$ or $x=6, y=17$.

Write:

$$\left.\begin{array}{ccccc} x & & y & & x \\ 1 & \times & 3 & — & 3 \\ 1 & & 3 & — & 3 \end{array}\right\} \text{ add: } x=6,\ y=17$$

$$\left.\begin{array}{ccccc} 1 & \times & 3 & — & 18 \\ 6 & & 17 & — & 17 \end{array}\right\} \text{ add: } x=35,\ y=99$$

...

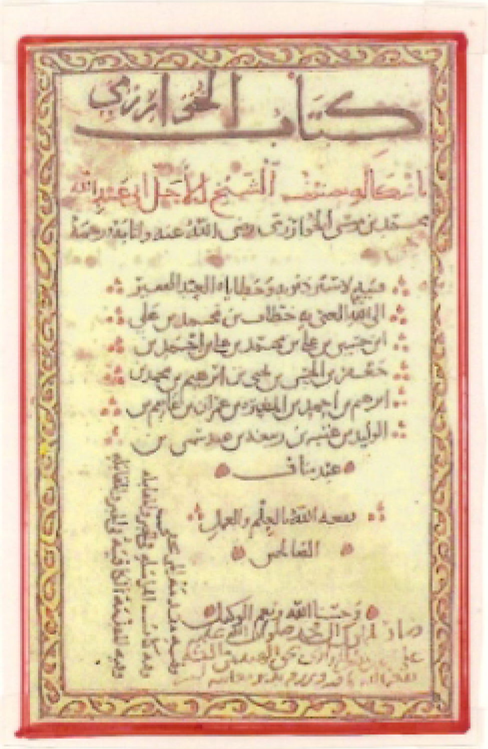
In general, solve $Cx^2 + 1 = y^2$:

$C = 67$: $\quad 67x^2 + 1 = y^2$

Solution: $x = 5967,\ y = 48842$

al-Khwarizmi (c. 825 AD)

Muhammad ibn Musa al-Khwarizmi


كتاب الخوارزمي

- Arithmetic text
- Algebra text

Kitab al-jabr w'al-muqabalah

= Ludus algebrae et almucgrabalaeque

Algorithmi de numero Indorum

'Dixit Algorismi'

Solving Quadratic Equations

Six types: (a, b, c all positive)

$ax^2 = bx$, $ax^2 = b$, $ax = b$,

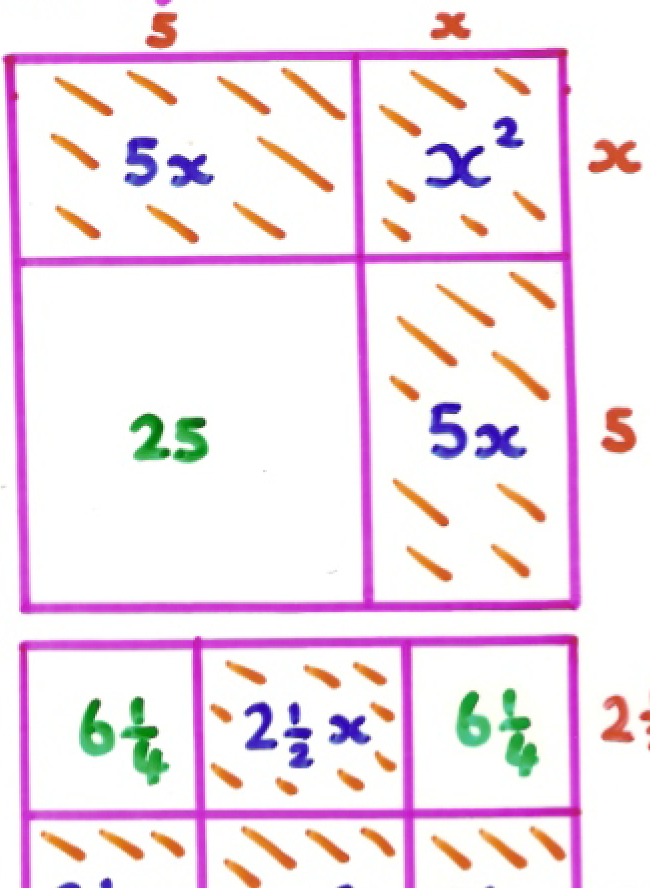
$ax^2 + bx = c$, $ax^2 + c = bx$, $ax^2 = bx + c$

'Roots and squares are equal to numbers'

$x^2 + 10x = 39$

$(x+5)^2 = 39 + 25 = 64$

so $x + 5 = 8$, $\underline{x = 3}$



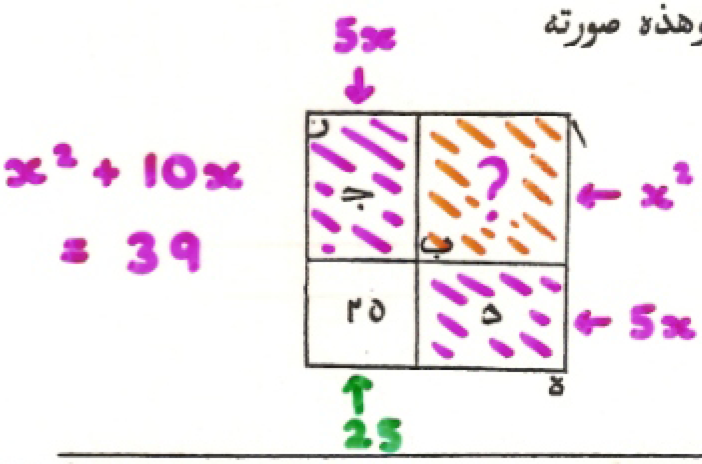
$x^2 + 10x = 39$

$(x + 2\frac{1}{2} + 2\frac{1}{2})^2$

$= 39 + (4 \times 6\frac{1}{4}) = 64$

so $x + 5 = 8$, $x = 3$

علي تسعة وثلثين ليتم السطح الاعظم الذي هو سطح ره فبلغ ذلك كله اربعة وستين فاخذنا جذرها وهو ثمانية وهو احد اضلاع السطح الاعظم فاذا نقصنا منه مثل ما زدنا عليه وهو خمسة بقي ثلثة وهو ضلع سطح اب الذي هو المال وهو جذره والمال تسعة وهذه صورته



واما مال واحد وعشرون درهما يعدل عشرة اجذاره فانا نجعل المال سطحا مربعا مجهول الاضلاع وهو سطح اد ثم نضم اليه سطحا متوازي الاضلاع عرضه مثل احد اضلاع سطح اد وهو ضلع هن والسطح هب فصار طول السطحين جميعا ضلع جه وقد علمنا ان طوله عشرة من العدد لان كل سطح مربع متساوي الاضلاع والزوايا فان احد اضلاعه مضروبا في واحد جذر ذلك السطح وفي اثنين جذراه فلما قال مال واحد وعشرون يعدل عشرة اجذاره علمنا ان طول ضلع هج عشرة اعداد لان ضلع جد جذر المال فقسمنا ضلع جه بنصفين علي نقطة

al-Khwarizmi's 'Algebra'

Omar Khayyam (c. 1100)

Equations in general:

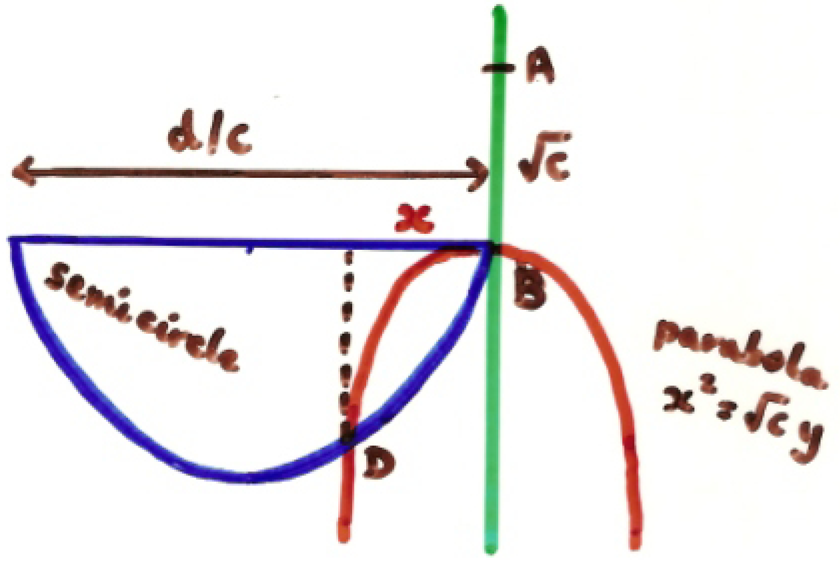
roots, squares, cubes,

square squares, square cubes, ...

A solid cube plus squares plus edges equal to a number ($x^3 + ax^2 + bx = c$)

Also: $x^3 + bx = c$, $x^3 = bx^2 + d$, ... [14 in all]

$x^3 + cx = d$

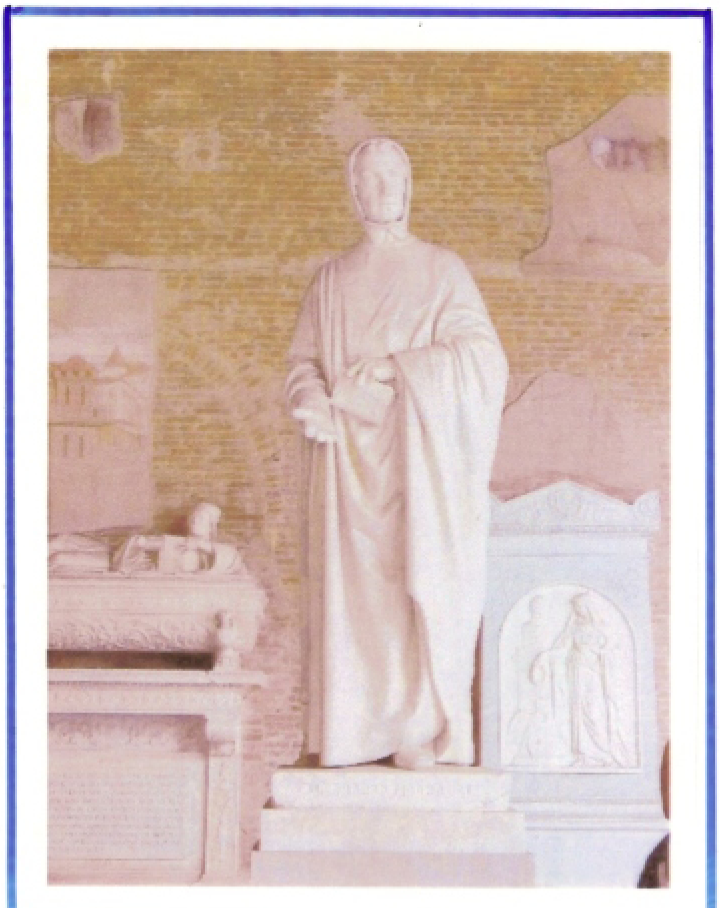


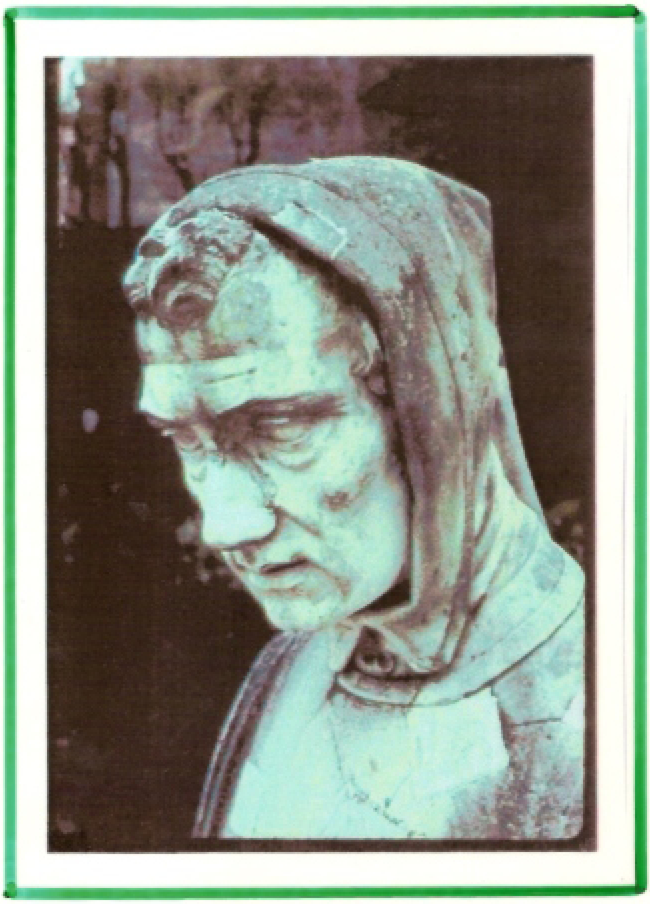
Fibonacci (Leonardo of Pisa)

Liber abaci (1202)

Book of squares

Hindu-Arabic numerals

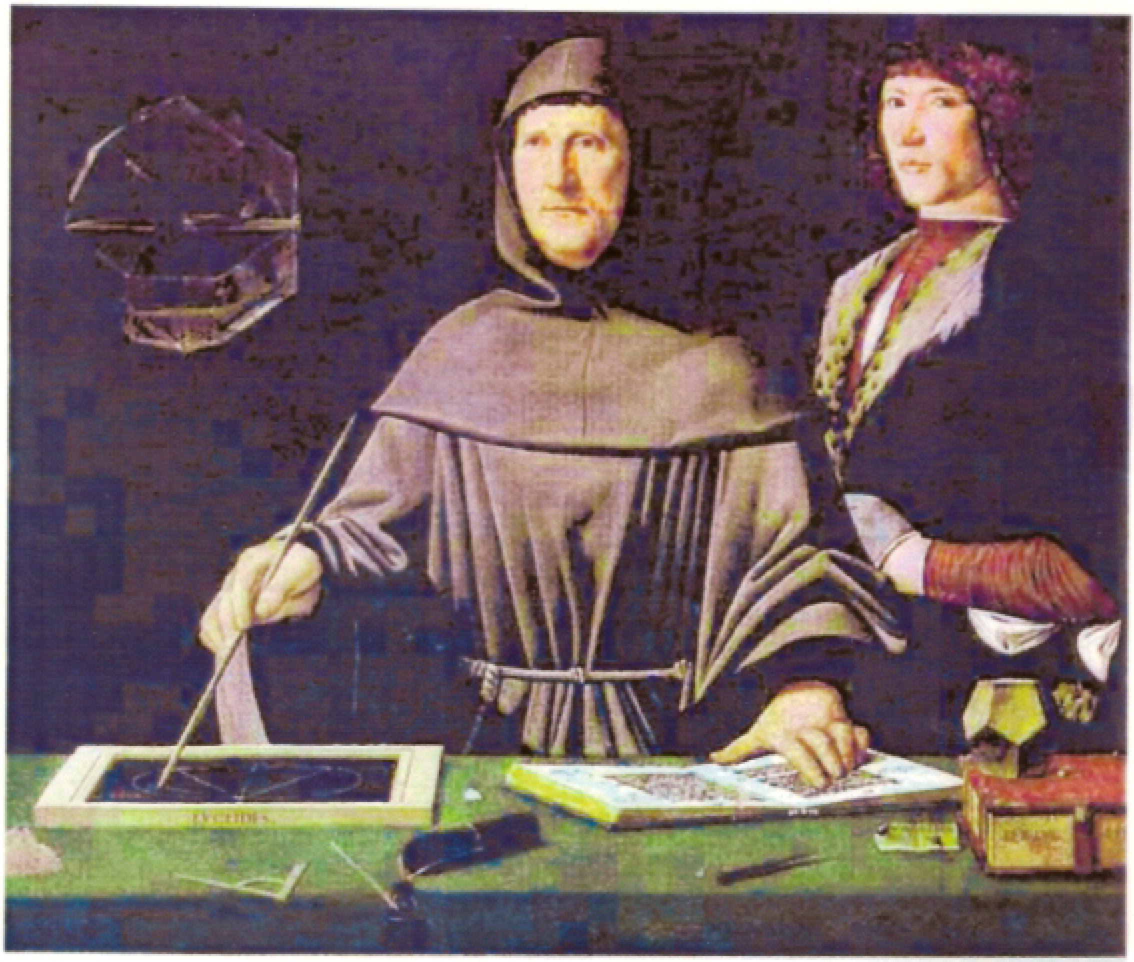


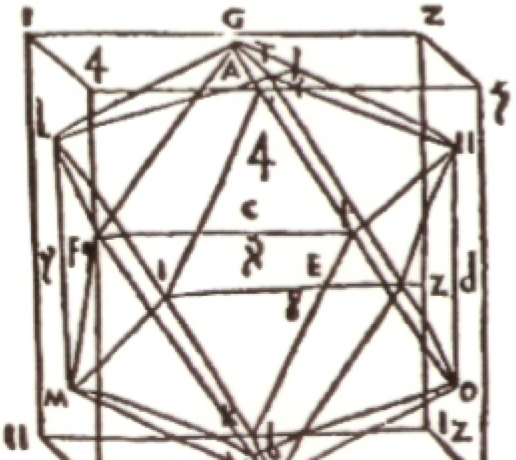


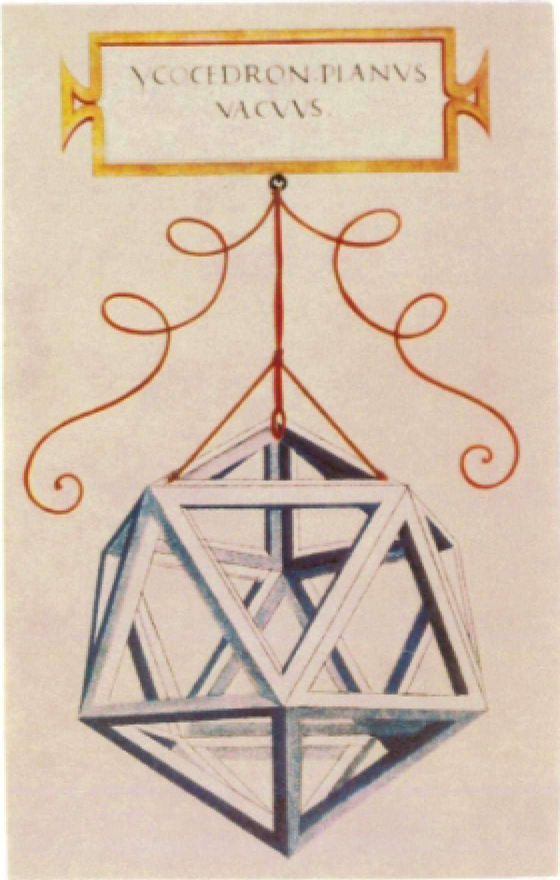
'Liber Abaci' - three problems

- There is a tree, $\frac{1}{4}$ and $\frac{1}{3}$ of which lie below ground; 21 palmi. How tall is the tree?
  Guess 12, giving $3 + 4 = 7$ (instead of 21): scale up to give $3 \times 12 = 36$.
- If I buy 3 sparrows for a penny, turtle-doves 2 for a penny, and doves for two pence - and spend 30 pence for 30 birds, how many of each kind do I buy?
- If a lion eats a sheep in 4 hours, a leopard eats it in 5 hours, and a bear eats it in 6 hours, how long would they all take together?

Luca Pacioli







icosahedron
in a cube


IN MEMORY OF
ROBERT RECORDE,
THE EMINENT MATHEMATICIAN,
WHO WAS BORN AT TENBY, CIRCA 1510.
TO HIS GENIUS WE OWE THE EARLIEST
IMPORTANT ENGLISH TREATISES ON
ALGEBRA, ARITHMETIC, ASTRONOMY, AND GEOMETRY:
HE ALSO INVENTED THE SIGN OF
EQUALITY = NOW UNIVERSALLY ADOPTED
BY THE CIVILIZED WORLD.
ROBERT RECORDE
WAS COURT PHYSICIAN TO
KING EDWARD VI. AND QUEEN MARY.
HE DIED IN LONDON,
1558.

The whetstone of witte,

whiche is the seconde parte of Arithmetike: containyng thextraction of Rootes: The Cossike practise, with the rule of Equation: and the woorkes of Surde Nombers.

Though many stones doe beare greate price,
The whetstone is for exersice
As neadefull, and in woorke as straunge:
Dulle thinges and harde it will so chaunge,
And make them sharpe, to right good vse:
All artesmen knowe, thei can not chuse,
But vse his helpe: yet as men see,
Noe sharpenesse semeth in it to bee.
The grounde of artes did brede this stone:
His vse is greate, and moare then one.
Here if you list your wittes to whette,
Moche sharpenesse therby shall you gette.
Dulle wittes hereby doe greately mende,
Sharpe wittes are fined to their fulle ende.
Now proue, and praise, as you doe finde,
And to your self be not vnkinde.

¶ These Bookes are to bee solde, at the Weste doore of Poules, by Jhon Kyngstone.

Whetstone of Witte (1557)

as their woorkes doe extende) to distincte it onely into twoo partes. Whereof the firste is, *when one nomber is equalle unto one other*. And the seconde is, *when one nomber is compared as equalle unto .2. other nombers*.

Alwaies willyng you to remeber, that you reduce your nombers, to their leaste denominations, and smalleste formes, before you procede any farther.

And again, if your *equation* be soche, that the greateste denomination *Cosike*, be ioined to any parte of a compounde nomber, you shall tourne it so, that the nomber of the greateste signe alone, maie stande as equalle to the reste.

And this is all that neadeth to be taughte, concernyng this woorke.

Howbeit, for easie alteratiō of *equations*. I will propounde a fewe exāples, bicause the extraction of their rootes, maie the more aptly bee wroughte. And to auoide the tediouse repetition of these woordes: is equalle to: I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe lines of one lengthe, thus: =====, bicause noe .2. thynges, can be moare equalle. And now marke these nombers.

1. 14.ze. —+— .15.ϑ ===== 71.ϑ.
2. 20.ze. ——— .18.ϑ ===== .102.ϑ.
3. 26.z. —+— 10ze ===== 9.z. ——— 10ze —+— 213.ϑ.
4. 19.ze —+— 192.ϑ ===== 10z —+— 108ϑ ——— 19ze
5. 18.ze —+— 24.ϑ. ===== 8.z. —+— 2.ze.
6. 34z ——— 12ze ===== 40ze —+— 480ϑ ——— 9.z.

1. In the firste there appeareth. 2. nombers, that is

14.ze.

Cubic equations

A cube + squares + edges = a number

$$x^3 + ax^2 + bx = c$$

A cube + things = numbers

$$x^3 + cx = d$$

Cubes + squares = numbers

$$ax^3 + bx^2 = d$$

$\left.\begin{array}{l}\text{Scipione del Ferro}\\ \text{Antonio Fiore}\end{array}\right\}\ x^3 + cx = d$

$\left.\text{Niccolo Tartaglia}\right\}\ x^3 + bx^2 = d$

Gerolamo Cardano

Rafael Bombelli

Fior's challenge to Tartaglia (1535)

1. Find me a number such that when its cube root is added to it, the result is 6.

$[x^3 + x = 6]$

...

15. A man sells a sapphire for 500 ducats, making a profit of the cube root of his capital. How much is the profit?

$[x^3 + x = 500]$

...

17. A tree, 12 *braccia* high, was broken into two parts at such a point that the height of the part that was left standing was the cube root of the length of the part that was cut away. What was the height of the part left standing?

$[x^3 + x = 12]$

...

30. There are two bodies of 20 triangular faces whose corporeal areas together make 700 *braccia* and the area of the smaller is the cube root of the larger. What is the smaller area?

$[x^3 + x = 700]$

QVESITI, ET INVENTIONI DIVERSE

DE NICOLO TARTAGLIA,

Di nouo restampati con vna Gionta al sesto libro, nella quale si mostra duoi modi di redur vna Città inespugnabile.

La diuisione, & continentia di tutta l'opra nel seguente foglio si trouarà notata.

... to enable me to remember the method

in any unforeseen circumstance,

I have arranged it as a verse in rhyme ...

When the cube and the thing together

Are equal to some discrete number,

$$x^3 + cx = d$$

Find two other numbers differing in this one.

$$u - v = d$$

Then you will keep this as a habit

That their product shall always be equal

Exactly to the cube of a third of the things.

$$uv = (c/3)^3$$

The remainder then as a general rule

Of their cube roots subtracted

Will be equal to this principal thing.

$$x = \sqrt[3]{u} - \sqrt[3]{v}$$

Solving a cubic equation

$$x^3 + 6x = 20$$

Find u and v so that

$$u - v = 20 \quad \text{and} \quad uv = (6/3)^3 = 8.$$

Since $v = u - 20$, we have

$$uv = u(u-20) = u^2 - 20u = 8.$$

Solving this quadratic equation:

$$u = \sqrt{108} + 10.$$

So $$v = u - 20 = \sqrt{108} - 10.$$

So

$$x = \sqrt[3]{u} - \sqrt[3]{v}$$

$$= \underline{\sqrt[3]{(\sqrt{108} + 10)} - \sqrt[3]{(\sqrt{108} - 10)}}$$

$$= 2.$$

HIERONYMI CARDANI, PRÆSTANTISSIMI MATHEMATICI, PHILOSOPHI, AC MEDICI,

ARTIS MAGNAE, SIVE DE REGVLIS ALGEBRAICIS,

Lib. unus. Qui & totius operis de Arithmetica, quod OPVS PERFECTVM inscripsit, est in ordine Decimus.

HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cossa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta euaserint. Neq; solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni ęquales fuerint, nodum explicant. Hunc aût librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmeticæ thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarētur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.

Cardano's Oath

To Tartaglia:

I swear to you, by God's holy Gospels, and as a true man of honour, not only never to publish your discoveries, if you teach me them, but I also promise you, and I pledge my faith as a true Christian, to note them down in code, so that after my death, no-one will be able to understand them.

Solving a quartic equation

$$x^4 = px^2 + qx + r$$

$$(x^2+y)^2 = (p+\underline{2y})\underline{x^2} + qx + (r+\underline{y^2})$$

For RHS a square we need:

$$(p+2y)(r+y^2) = q^2/4 \qquad [\text{“}b^2 = 4ac\text{”}]$$

$$(*) \qquad 2y^3 + py^2 + 2ry + (pr - q^2/4) = 0$$

$(*)$ then reduces to two quadratics.

e.g. $$x^4 = x + 2$$

$$x = \sqrt{\sqrt[3]{\sqrt{\frac{2075}{442368}} + \frac{1}{128}} - \sqrt[3]{\sqrt{\frac{2075}{442368}} - \frac{1}{128}}}$$

$$+ \sqrt{\sqrt[3]{\frac{1051}{3456} + \sqrt{\frac{2075}{442368}}} + \sqrt[3]{\frac{1051}{3456} - \sqrt{\frac{2075}{442368}}} \cdots}$$

$$\overline{\cdots + \frac{2}{3} - \sqrt[3]{\sqrt{\frac{2075}{442368}} + \frac{1}{128}} - \sqrt[3]{\sqrt{\frac{2075}{442368}} - \frac{1}{128}}}$$

Cardano's problem

Divide 10 into two parts whose product is 40.

If the parts are x and $10-x$, then $x(10-x) = 40$.

Cardano: $x = 5+\sqrt{-15}$ or $5-\sqrt{-15}$

'Nevertheless we will operate, putting aside the mental tortures involved.'

$$x(10-x) = (5+\sqrt{-15})(5-\sqrt{-15})$$

$$= 25-(-15) = 40.$$

L'ALGEBRA
OPERA

Di RAFAEL BOMBELLI da Bologna
Diuisa in tre Libri.

Con la quale ciascuno da se potrà venire in perfetta cognitione della teorica dell'Arimetica.

Con vna Tauola copiosa delle materie, che in essa si contengono.

Posta hora in luce à beneficio delli studiosi di detta professione.

IN BOLOGNA,
Per Giouanni Rossi. MDLXXIX.
Con licenza de' Superiori

Bombelli and complex numbers

$$x^3 = 15x + 4$$

solutions: $4, -2+\sqrt{3}, -2-\sqrt{3}$

Cardano's method yields

$$x = \sqrt[3]{(2+\sqrt{-121})} - \sqrt[3]{(-2+\sqrt{-121})}$$

first appearance of complex numbers

Bombelli calculated that:

$$(2+\sqrt{-1})^3 = 2+\sqrt{-121},$$

$$(2-\sqrt{-1})^3 = 2-\sqrt{-121}.$$

So $x = (2+\sqrt{-1}) - (-2+\sqrt{-1}) = 4.$

Ancora di

Ame ce si hauesse ad agguagliare 1 a 15 p

che i 5 cubati fa 125. et questo si caua del qua

4: restarà 0 m. 121, che di questo pigliata la

giunta

1 Eguale à 15 p 4 2 p R

ator Cu

5 / 4 2 / 4

fara R[2 p R

125 / 125 125

et tanto uale l

R[0 m 121] possa piu

com

Sõma R[0 m 121] Caua R[0 m 121]

R[2 p. R[0 m. 121]] p R[2 m R[0 m. 121]]

Creatore 2 p R[0 m 1] p. 2 m. R[0 m. 1]

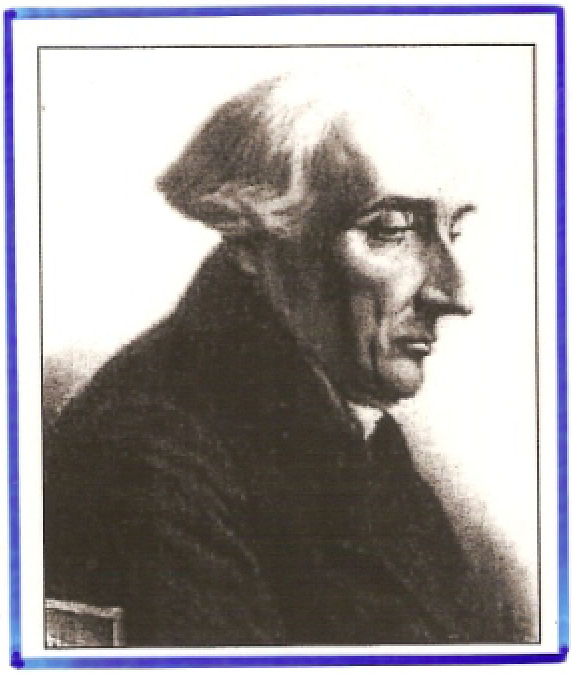
2 m. R[0 m. 1]

Somma 4: et tanto uale la Cosa

R[2 p R[0 m 121]] sara 2 p R[0 m. 1] che

2 m R[0 m. 1] che aggiunti insieme fanno 4.

Lagrange's reduction methods



$$x^3 + nx + p = 0$$

put $x = y - \frac{n}{3y}$

$$y^6 + py^3 - \frac{1}{27}n^3 = 0$$

quadratic in y^3

six $\underline{y}$s → three $\underline{x}$s

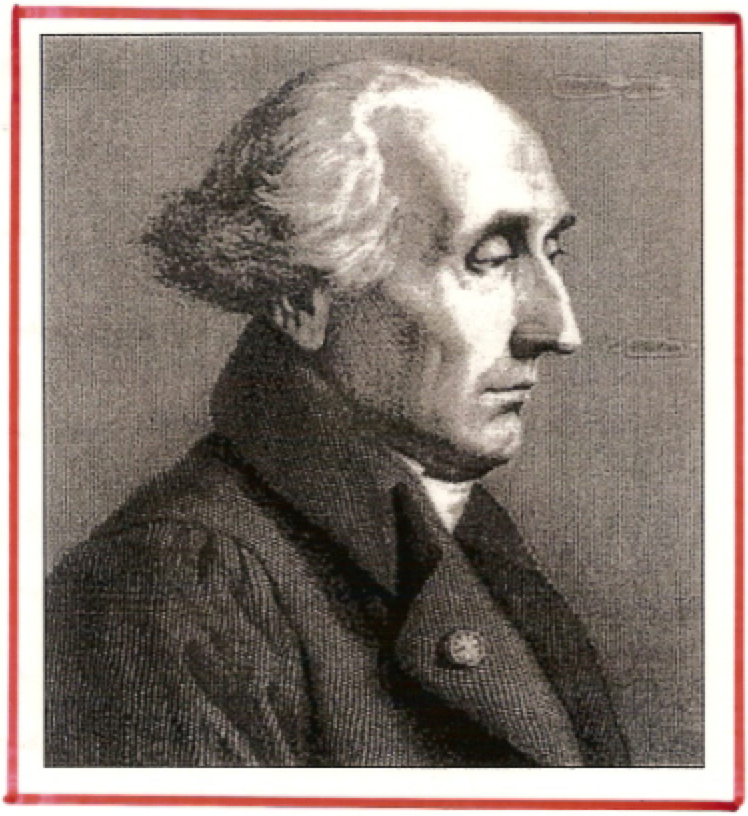
Quartic: $x^4 + nx^2 + px + q = 0$

can be reduced to

$$y^3 - \frac{n}{2}y^2 - qy + \frac{4nq - p^2}{8} = 0$$

cubic in y, three $\underline{y}$s → twelve $\underline{x}$s

→ four different $\underline{x}$s



Permuting the solutions

$$x^3 + ax^2 + bx + c = 0$$

solutions p, q, r

$$x^3 + ax^2 + bx + c = (x-p)(x-q)(x-r)$$

$c = -pqr$, $b = pq + pr + qr$,
$a = -(p+q+r)$

Permute the solutions – no change

$pq + 2r$: $pr + 2q$ $qr + 2p$ (3 values)

$p + 2q + 5r$: $p + 2r + 5q, \ldots$ (6 values)

… (always divides 6)

$(p-q)(q-r)(r-p)$ (2 values)

Paolo Ruffini

(1765 - 1822)

The algebraic solution of equations of degree greater than 4 is always impossible.

Behold a very important theorem which I believe I am able to assert ... to present the proof of it is the main reason for publishing this volume ...

Niels Henrik Abel

(1802 - 1829)

proved impossibility of solving quintic equations

Evariste Galois

(1811 - 1832)

developed criteria for deciding which equations can be solved

Galois to Chevalier, 29 May 1832

My dear friend,

I have done several new things in analysis. Some concern the theory of equations; others, integral functions.

In the theory of equations I have found out in which cases the equations are solvable by radicals, which has given me the occasion to deepen the theory and to describe all the transformations admitted by an equation, even when it is not solvable by radicals.

. . .

You will publicly beg Jacobi or Gauss to give their opinion not of the truth but of the importance of the theorems.

After this, there will, I hope, be people who will find it to their advantage to decipher all this mess.

Four bells : Plain Bob Minimus

1 2 3 4
e
2 1 4 3
o
2 4 1 3
e
4 2 3 1
o
4 3 2 1
e
3 4 1 2
o
3 1 4 2
e
1 3 2 4
|
1 3 4 2
e
3 1 2 4
o
3 2 1 4
e
2 3 4 1
o
2 4 3 1
e
4 2 1 3
o
4 1 2 3
e
1 4 3 2
|
1 4 2 3
e
4 1 3 2
o
4 3 1 2
e
3 4 2 1
o
3 2 4 1
e
2 3 1 4
o
2 1 3 4
e
1 2 4 3
|
1 2 3 4

Interchange consecutive bells :

1 2 3 4

(even)

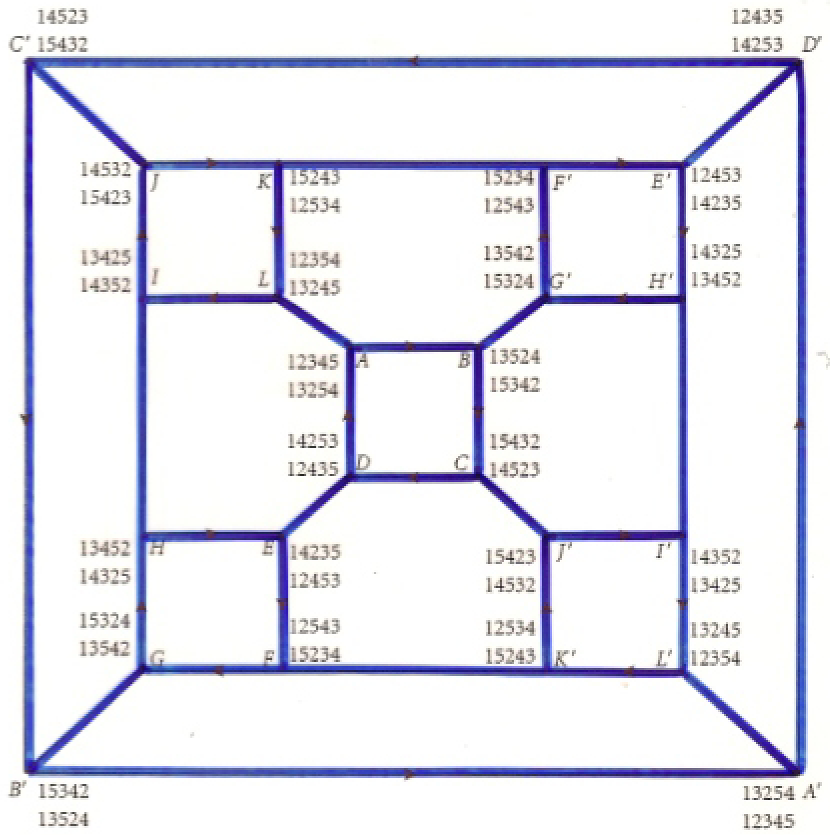
2 1 4 3

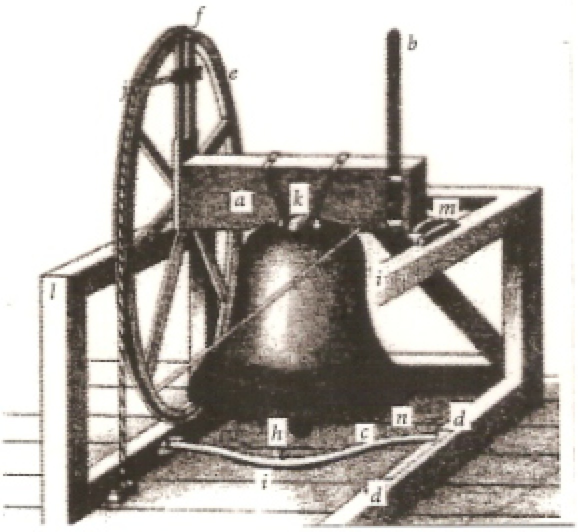
(odd)

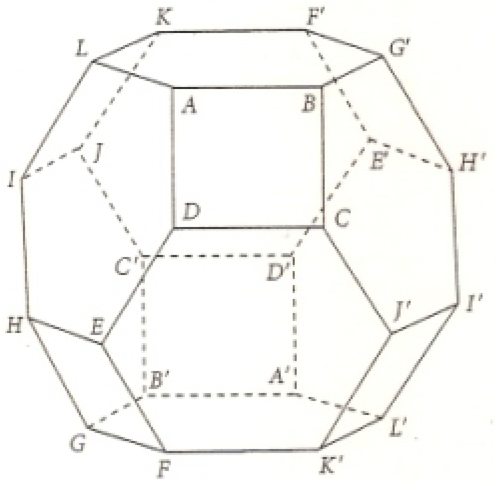
2 4 1 3

. . .

Five bells : Plain Bob Doubles

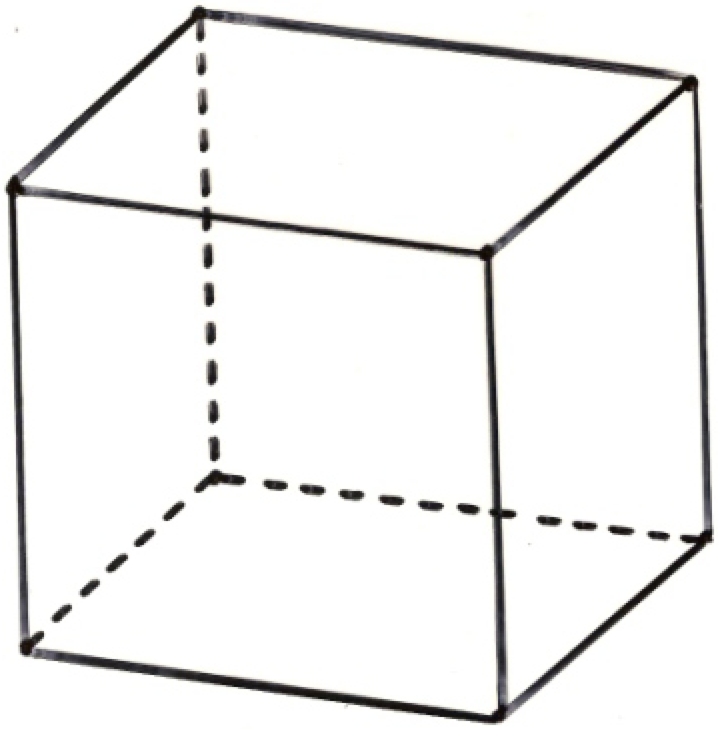






truncated octahedron

Symmetries of a Cube

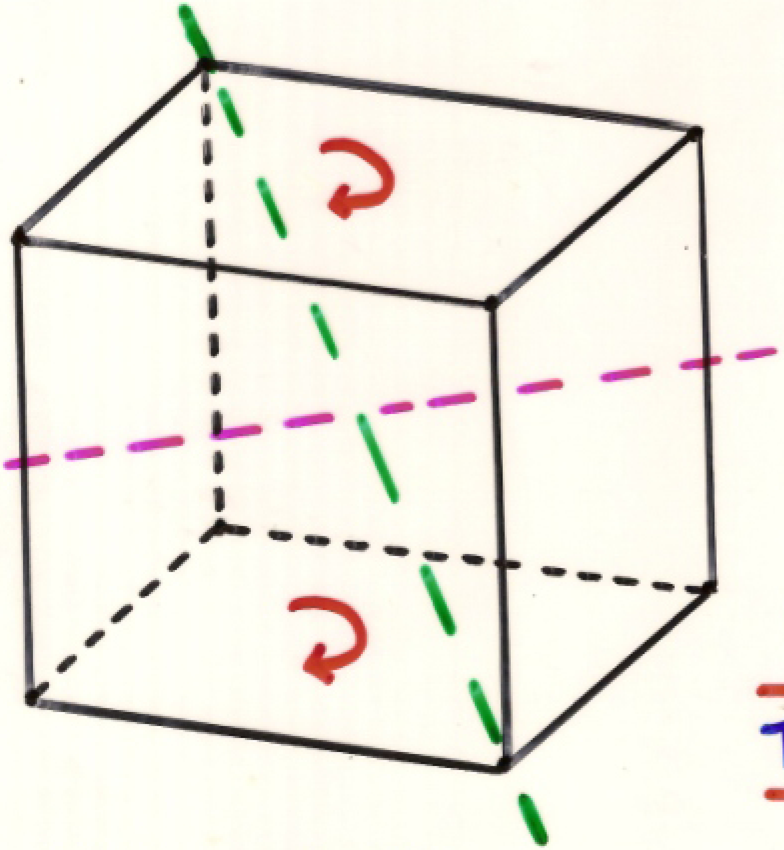


24 rotations

24 'indirect' symmetries

(reflection + rotation)

Symmetries of a Cube



Rotations

1 identity

9 face rotations

8 vertex rotns

6 edge rotns

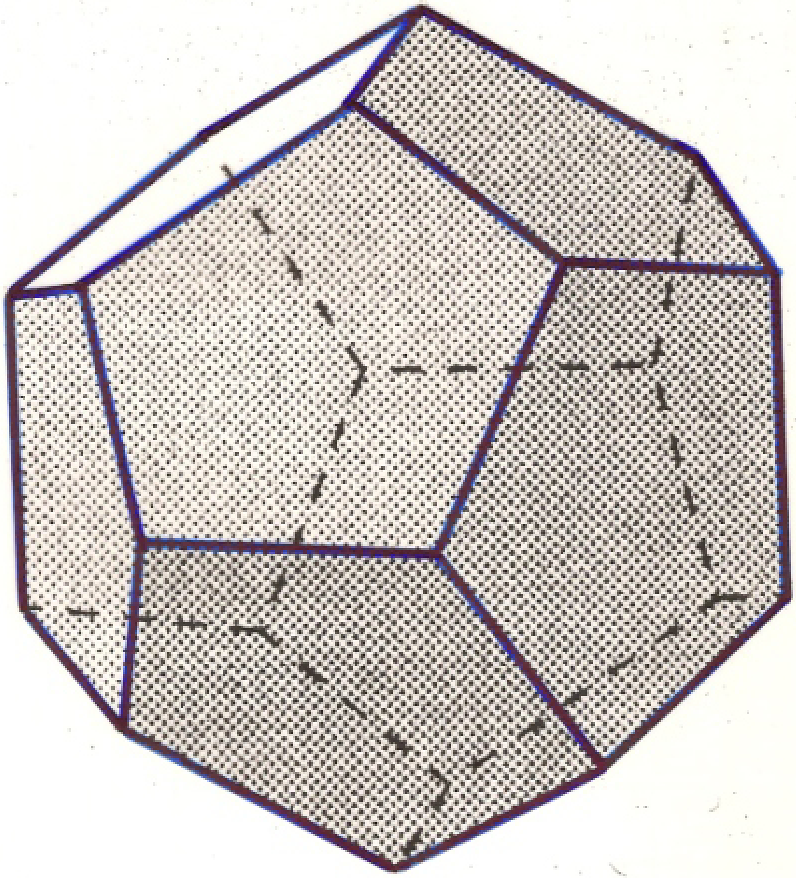
Total : 24

24 rotations

24 'indirect' symmetries

(reflection + rotation)

Rotations of a Dodecahedron



60 rotations

Abstract Groups

Take a set G of elements, and a rule for combining them ($*$). For a group, we have:

(1) Closure: if a and b are in G, then so is $a*b$.

(2) Associativity: if a, b, c are in G, then $(a*b)*c = a*(b*c)$

(3) Identity: there is an element e such that $a*e = e*a = a$ for all a.

(4) Inverses: for each a in G, there is an inverse a^{-1} such that $a*a^{-1} = e$.

[(5) Abelian: if a and b are in G, then $a*b = b*a$.]

Remembering the group axioms

C losure

A ssociative

I nverses

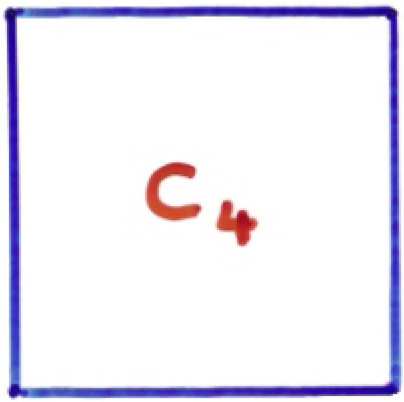
N eutral element

not necessarily **ABEL**ian

Examples :

- permutations of 4 bells
- symmetries of a cube
- rotations of a dodecahedron
- addition of integers
- multiplying positive real numbers

Cyclic Groups

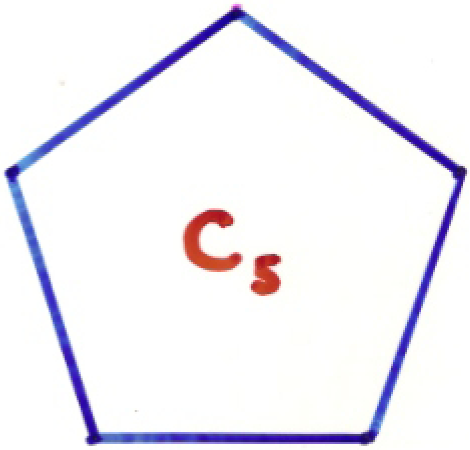


rotations of a square through 0, 1, 2, 3 right angles:

$1+3=0$, $3+2=1$, ...
(arithmetic 'mod 4')

rotations of a pentagon through 0, 1, 2, 3, 4 turns:

$2+3=0$, $3+4=2$, ...



Classification theorem: Every abelian group is obtained by combining cyclic groups.

Simple groups

- 'building blocks' for groups in general
- groups that 'contain no other groups inside them' – they have no proper normal subgroups

- cyclic groups with a prime number of elements : C_5, C_7, C_{101}, ...
- rotations of a dodecahedron (A_5)
- A_n ($n \geq 5$) : even permutations of $\{1, 2, \cdots, n\}$
- groups of 'Lie type' (related to matrices)
- 26 'sporadic groups'

[Monster group : rotations in 196883-dimensional space]