

4000 YEARS OF ALGEBRA

Robin Wilson

The Open University

x

C

N

q

$$x + \frac{1}{2}x = 16$$

y^2

Z

$\sqrt{d}$

a^3

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$e \cdot x = x$$

b

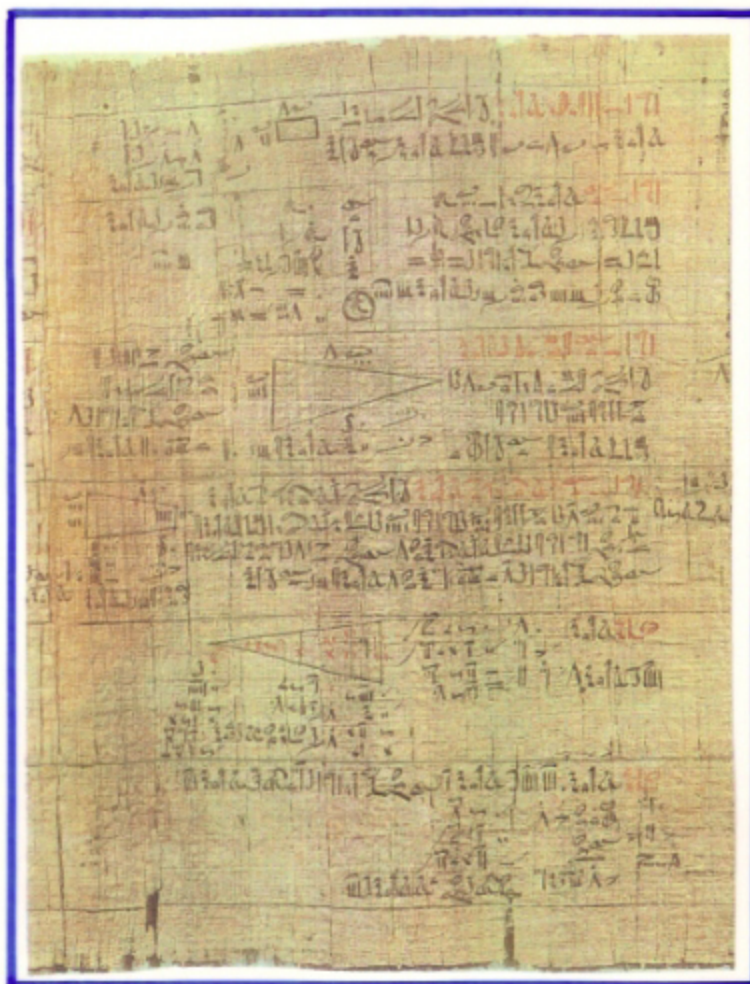
x

$$\sqrt{b^2 - 4ac}$$

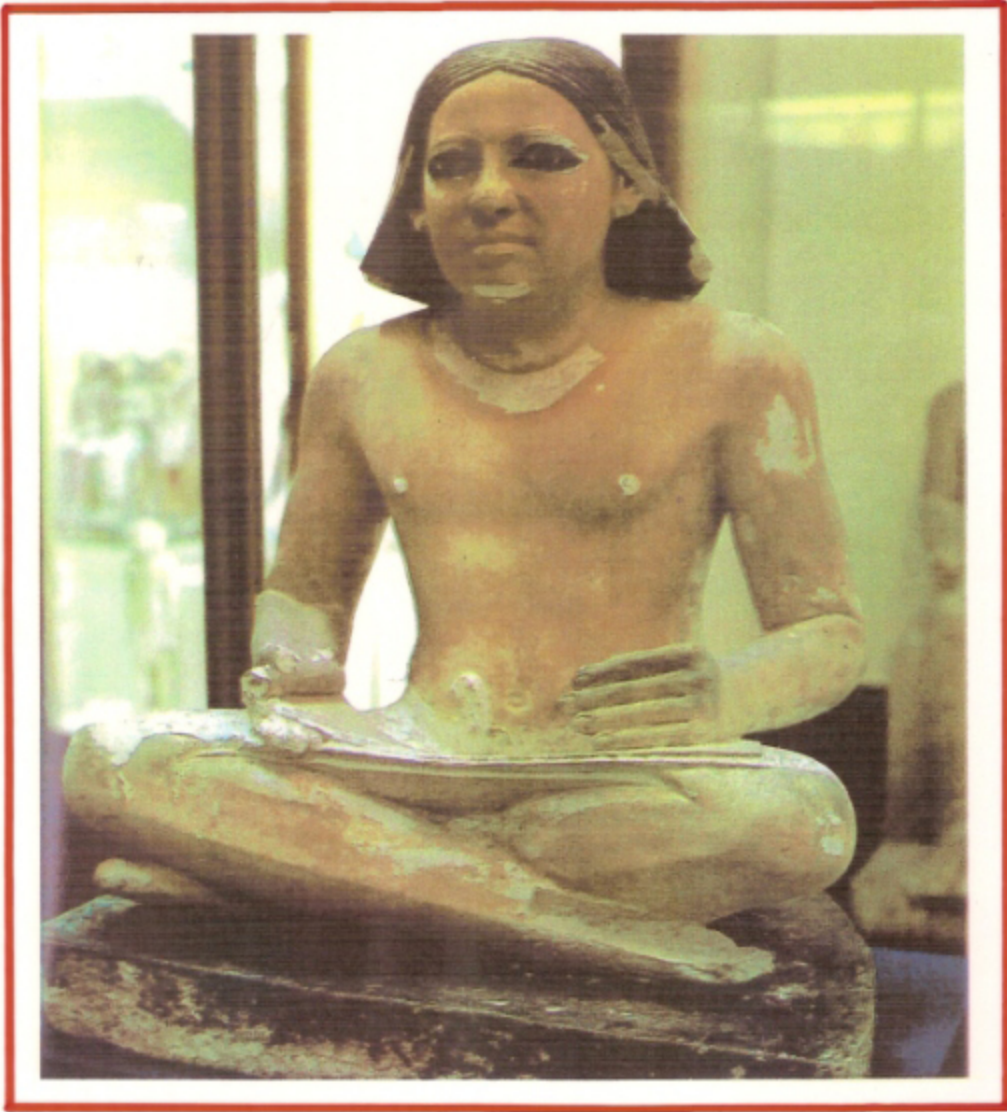
y

Rhind Papyrus

(c. 1650 BC)



Egyptian Scribe



Problem 25

A quantity and its $\frac{1}{2}$ added together become 16.

What is the quantity?

$$x + \frac{1}{2}x = 16$$

Assume 2

1	2
$\frac{1}{2}$	1
Total	3

method of
false
position

As many times as 3 must be multiplied to give 16,
so many times 2 must be multiplied to give the
required number.

1	3	← 16
2	6	
4	12	
$\frac{2}{3}$	2	
$\frac{1}{3}$	1	
Total	$5\frac{1}{3}$	

1	$5\frac{1}{3}$
2	$10\frac{2}{3}$

Do it thus:

The quantity is $10\frac{2}{3}$

$$\frac{1}{2} \quad 5\frac{1}{3}$$

$$\text{Total} \quad 16$$

Egyptian fractions

Unit fractions:

(reciprocals)

$$\frac{1}{n} \text{ (and } \frac{2}{3})$$

$$\frac{2}{11} = \frac{1}{6} + \frac{1}{66}$$

$$\frac{2}{13} = \frac{1}{8} + \frac{1}{52} + \frac{1}{104}$$

Rhind papyrus, Problem 31

A quantity, its $\frac{2}{3}$, its $\frac{1}{2}$, and its $\frac{1}{7}$,
added together, become 33.

What is the quantity?

$$[\text{Solve: } x + \frac{2}{3}x + \frac{1}{2}x + \frac{1}{7}x = 33]$$

Solution: The total is

$$14 \frac{1}{4} + \frac{1}{56} + \frac{1}{97} + \frac{1}{194} + \frac{1}{388} + \frac{1}{679} + \frac{1}{776},$$

$$14 \frac{28}{97}$$

which multiplied by $1 \frac{2}{3} \frac{1}{2} \frac{1}{7}$ makes 33.

Table of Fractions

$2/n$, for $n = 5, 7, 9, 11, \dots, 99, 101$

The image shows a fragment of an ancient papyrus scroll, likely from the Rhind Mathematical Papyrus, containing a table of fractions. The text is written in hieroglyphs and is organized into columns and rows, with some entries highlighted in red ink. The table lists the values of $2/n$ for various values of n , expressed as sums of unit fractions. The fragment is framed by a blue border.

n	$2/n$
5	$2/5 = 1/3 + 1/15$
7	$2/7 = 1/4 + 1/28$
9	$2/9 = 1/5 + 1/45$
11	$2/11 = 1/6 + 1/66$
13	$2/13 = 1/7 + 1/91$
15	$2/15 = 1/8 + 1/120$
17	$2/17 = 1/9 + 1/153$
19	$2/19 = 1/10 + 1/190$
21	$2/21 = 1/11 + 1/231$
23	$2/23 = 1/12 + 1/276$
25	$2/25 = 1/13 + 1/325$
27	$2/27 = 1/14 + 1/378$
29	$2/29 = 1/15 + 1/435$
31	$2/31 = 1/16 + 1/496$
33	$2/33 = 1/17 + 1/561$
35	$2/35 = 1/18 + 1/630$
37	$2/37 = 1/19 + 1/703$
39	$2/39 = 1/20 + 1/780$
41	$2/41 = 1/21 + 1/858$
43	$2/43 = 1/22 + 1/946$
45	$2/45 = 1/23 + 1/1035$
47	$2/47 = 1/24 + 1/1152$
49	$2/49 = 1/25 + 1/1225$
51	$2/51 = 1/26 + 1/1326$
53	$2/53 = 1/27 + 1/1431$
55	$2/55 = 1/28 + 1/1540$
57	$2/57 = 1/29 + 1/1653$
59	$2/59 = 1/30 + 1/1770$
61	$2/61 = 1/31 + 1/1886$
63	$2/63 = 1/32 + 1/2016$
65	$2/65 = 1/33 + 1/2145$
67	$2/67 = 1/34 + 1/2278$
69	$2/69 = 1/35 + 1/2415$
71	$2/71 = 1/36 + 1/2556$
73	$2/73 = 1/37 + 1/2685$
75	$2/75 = 1/38 + 1/2850$
77	$2/77 = 1/39 + 1/2982$
79	$2/79 = 1/40 + 1/3160$
81	$2/81 = 1/41 + 1/3282$
83	$2/83 = 1/42 + 1/3437$
85	$2/85 = 1/43 + 1/3595$
87	$2/87 = 1/44 + 1/3762$
89	$2/89 = 1/45 + 1/3960$
91	$2/91 = 1/46 + 1/4130$
93	$2/93 = 1/47 + 1/4347$
95	$2/95 = 1/48 + 1/4560$
97	$2/97 = 1/49 + 1/4788$
99	$2/99 = 1/50 + 1/4950$
101	$2/101 = 1/51 + 1/5151$

Mesopotamian Mathematics

clay tablets – cuneiform writing

place-value system based on 60: <, |

$$<<<|<<< = 41(60) + 40, \text{ or } 41\frac{40}{60}, \text{ or } \dots$$



Weighing a Stone

I found a stone, but did not weigh it;
after I weighed out 6 times its weight,
added 2 gin,
and added one-third of one-seventh
multiplied by 24,
I weighed it: 1 ma-na.

What was the original weight of the stone?

[Tablet had 22 such problems: 1 ma-na = 60 gin]

Solution:

$$(6x + 2) + \frac{1}{3} \cdot \frac{1}{7} \cdot 24 (6x + 2) = 60 \text{ gin}$$



so $x = 4\frac{1}{3} \text{ gin}$.

Check: $28 + \frac{1}{3} \cdot \frac{1}{7} \cdot 24 \cdot 28 = 28 + 32 = 60.$

Solving a 'Quadratic Equation'

I have subtracted the side of my square from the area: 14, 30.

You write down 1, the coefficient.

You break off half of 1. 0;30 and 0;30 you multiply. You add 0;15 to 14,30.

Result 14,30; 15. This is the square of 29;30.

You add 0;30, which you multiplied, to 29;30.

Result: 30, the side of the square.

$$\underline{x^2 - x = 870:}$$

$$1 \rightarrow \frac{1}{2} \rightarrow \left(\frac{1}{2}\right)^2 = \frac{1}{4} \rightarrow 870\frac{1}{4} \rightarrow 29\frac{1}{2} \rightarrow 30.$$

$$\underline{x^2 - bx = c:}$$

$$\begin{aligned} b \rightarrow \frac{b}{2} \rightarrow \left(\frac{b}{2}\right)^2 \rightarrow \left(\frac{b}{2}\right)^2 + c \rightarrow \sqrt{\left(\frac{b}{2}\right)^2 + c} \\ \rightarrow \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + c}. \end{aligned}$$

The igum and the igibum

The igibum exceeds the igum by 7.

What are the igum and the igibum?

$$x - y = 7$$

$$xy = 60$$

Halve 7, and the result is 3;30.

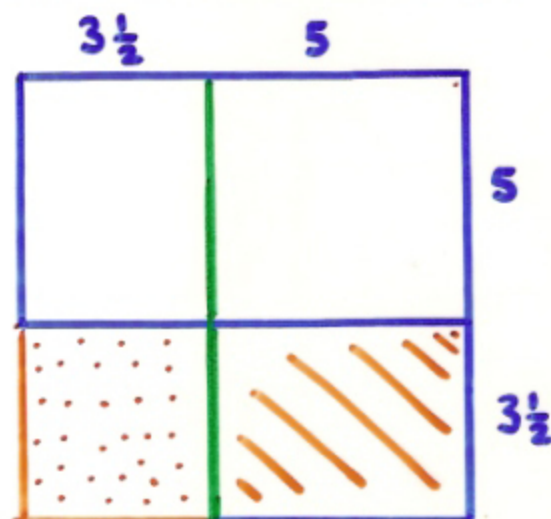
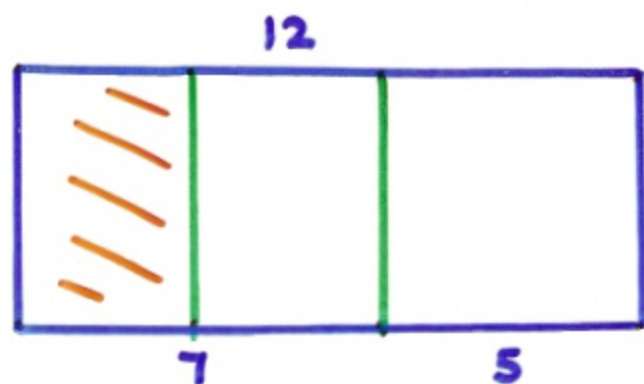
Multiply 3;30 with 3;30, and we get 12;15.

Add 1,0, the product, and we get 1,12;15.

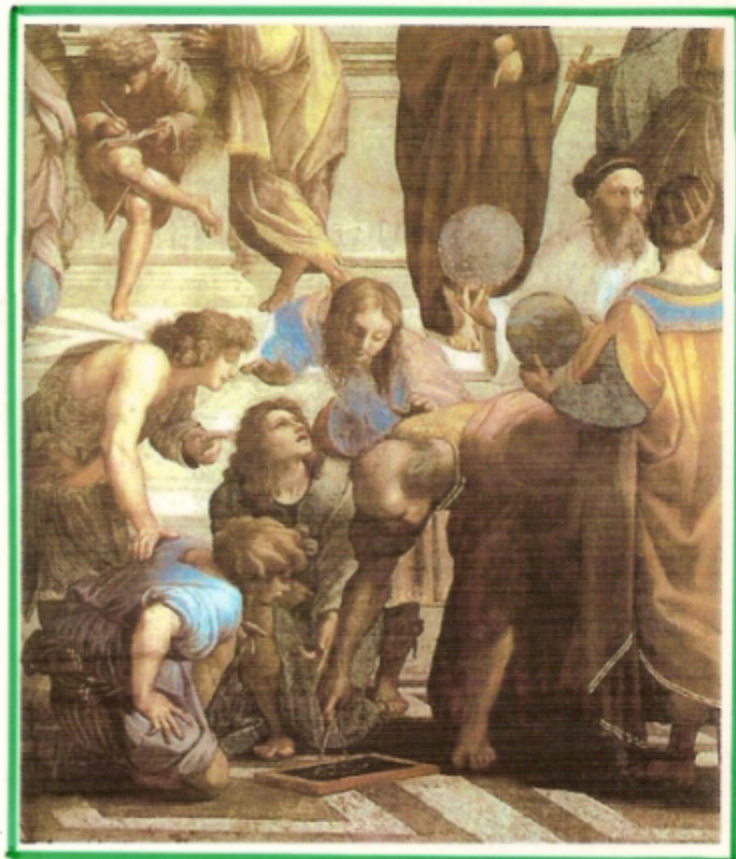
What is the square root of 1,12;15? 8;30

Lay down 8;30 and 8;30 and subtract 3;30 from one and add it to the other.

One is 12 (the igibum), the other 5 (the igum).



$$x, y = \sqrt{(3\frac{1}{2})^2 + 60} \pm 3\frac{1}{2} = 12 \text{ or } 5.$$



EVCLIDIS MEGAREN

SIS CLARISSIMI PHILOSOPHI, MATHEMATICORVM
facile principis: primum ex Campano, deinde ex Theone graeco com-
mentatore, interprete Bartholomaeo Zamberto Veneto,
Geometricorum elementorum liber primus.

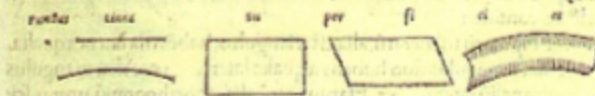
Ex Campano triplex principiorum genus.
Primum Diffinitiones.



Punctus est, cuius pars non est. 1. Linea, est
longitudo sine latitudine. 2. Cuius quidem
extremitates, sunt duo puncta. 3. Linea re-
cta, est ab uno puncto ad alium brevissima ex-
tensio; in extremitates suas eos recipiens.

4. Superficies, est quae longitudinem & lati-
tudinem tantum habet. 5. Cuius quidem ter-
mini, sunt lineae. 6. Superficies plana, est ab

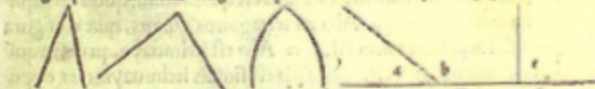
una linea ad aliam brevissima extensio, in extremitates suas eas recipiens.



7. Angulus planus, est duarum linearum alternus contactus, quarum
expansio est super superficiem, applicatioque non directa. 8. Quando
autem angulum continent duae lineae rectae rectilineus angulus nominatur.

9. Quando recta linea super rectam steterit, duoque anguli utrobique
sint aequales, eorum uterque rectus erit, lineaeque lineae superstant, ei cui su-
perstat, perpendicularis vocatur. 10. Angulus vero qui recto maior est,
obtusus dicitur. 11. Angulus vero minor recto, acutus appellatur.

Angulus planus rectilineus angulus obliquus acutus obtusus rectus perpendicularis et rectus



12. Terminus, est quod uniuscuiusque finis est. 13. Figura, est quae ter-
mino vel terminis continetur. 14. Circulus, est figura plana una qui-
dem linea contenta quae circumferentia nominatur, in cuius medio punctus
est, a quo omnes lineae rectae ad circumferentiam exeuntes, sibi inuicem



Euclid

(c. 300 BC)

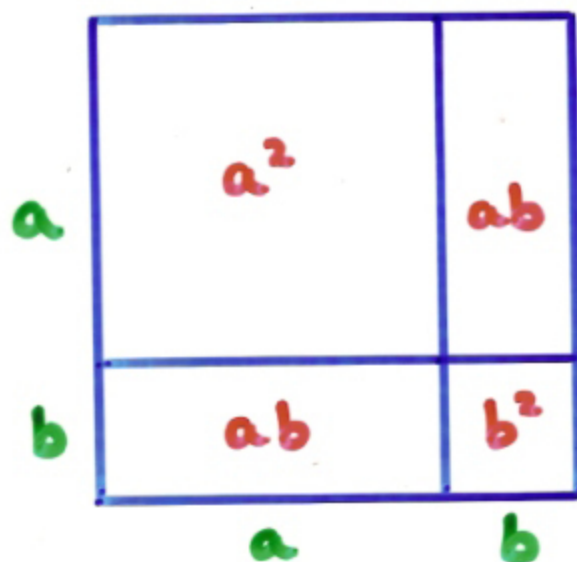
Euclid : Book II

II.1



$$a(b_1 + b_2 + b_3) = ab_1 + ab_2 + ab_3$$

II.4



$$(a+b)^2 = a^2 + b^2 + 2ab$$

How old was Diophantus?

Diophantus spent $\frac{1}{6}$ of his life in childhood, $\frac{1}{12}$ in youth, and $\frac{1}{7}$ more as a bachelor.

Five years after his marriage there was a son who died four years before his father at $\frac{1}{2}$ his father's final age.

x = Diophantus's age :

$$\left(\frac{1}{6}x + \frac{1}{12}x + \frac{1}{7}x \right) + 5 + \frac{1}{2}x + 4 = x,$$

so $x = 84$ years

DIOPHANTI

ALEXANDRINI ARITHMETICORVM

LIBRI SEX,
ET DE NVMERIS MVLTVGLIS

LIBER VNVS.

*CVM COMMENTARIIS C. G. BACHETI V. C.
& observationibus D. P. de FERMAT Senatoris Tolofani.*

Accessit Doctrinæ Analyticæ inuentum nouum, collectum
ex varijs eiusdem D. de FERMAT Epistolis.



TOLOSÆ,
Excudebat BERNARDVS BOSC, è Regione Collegij Societatis Iesu.

M. DC. LXX. M

Problems of Diophantus

Solutions in whole numbers or fractions:

$$2x + 3y = 10: \quad x = 2, y = 2$$

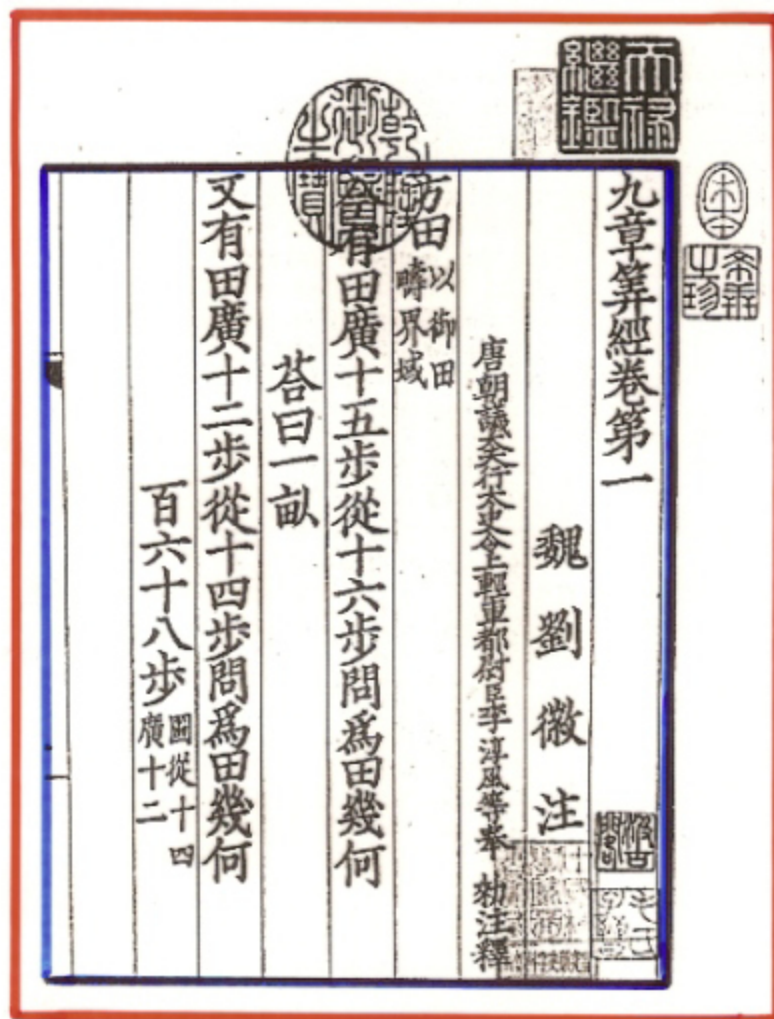
Find two square numbers, the sum of which is a cubic number:

$$25 + 100 = 125$$

Find a right-angled triangle such that its perimeter is a square, while its perimeter added to its area gives a cube.

$$\frac{1024}{217}, \frac{215055}{47089}, \frac{309233}{47089}$$

Jiuzhang suanshu



Opening of Chapter 1

agriculture, business, surveying, etc

- calculation of areas and volumes
- calculation of square/cube roots
- study of right-angled triangles
- simultaneous equations

Nine Chapters on the Mathematical Art

(200 BC?)

246 questions
with answers
(but no 'working')

A problem involving grain

Three types of grain: A, B and C.

3 bundles of A, 2 of B, 1 of C = 39 measures;

2 bundles of A, 3 of B, 1 of C = 34 measures;

1 bundle of A, 2 of B, 3 of C = 26 measures.

How many measures in one bundle of each type?

(1) $3A + 2B + 1C = 39$

(2) $2A + 3B + 1C = 34$

(3) $1A + 2B + 3C = 26$

I	II	III	(A)
II	III	II	(B)
III	I	I	(C)
= T	≡ IIII	≡ TTTT	
(3)	(2)	(1)	

1	2	3
2	3	2
3	1	1
26	34	39



0	0	3
0	5	2
36	1	1
99	24	39

$$\begin{aligned} & 36C = 99 \\ & \rightarrow 5B + C = 24 \\ & 3A + 2B + C = 39 \end{aligned} \rightarrow \begin{cases} C = 2\frac{3}{4} \\ B = 4\frac{1}{4} \\ A = 9\frac{1}{4} \end{cases}$$

The Chinese Remainder Theorem

Sun Zi (250 AD) in Sunzi suanjing

(Master Sun's mathematical manual).

We have things of which we do not know the number.

If we count them by 3s, the remainder is 2.

If we count them by 5s, the remainder is 3.

If we count them by 7s, the remainder is 2.

How many things are there?

$$N \equiv 2 \pmod{3} \equiv 3 \pmod{5} \equiv 2 \pmod{7}$$

The answer is 23.

Bhaskara (1100AD) : 'Pell's equation'

Tell me, O mathematician, what is that square which multiplied by 8 becomes – together with unity – a square?

$$8x^2 + 1 = y^2$$

Solutions : $x=1, y=3$ or $x=6, y=17$.

Write:

$$\begin{array}{ccc} x & y & x \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{array} \left. \vphantom{\begin{array}{ccc} x & y & x \\ 1 & 3 & 3 \\ 1 & 3 & 3 \end{array}} \right\} \text{add: } x=6, y=17$$

$$\begin{array}{ccc} 1 & 3 & 18 \\ 6 & 17 & 17 \end{array} \left. \vphantom{\begin{array}{ccc} 1 & 3 & 18 \\ 6 & 17 & 17 \end{array}} \right\} \text{add: } x=35, y=99$$

...

In general, solve $Cx^2 + 1 = y^2$:

$$C=67: \quad 67x^2 + 1 = y^2$$

$$\text{Solution: } x=5967, y=48842$$



al-Khwarizmi (c. 825AD)

Muhammad ibn Musa al-Khwarizmi

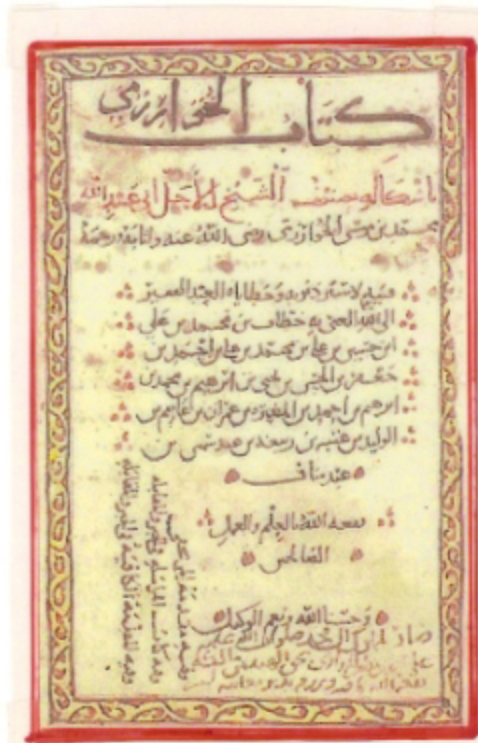
- Arithmetic text

- Algebra text

Kitab al-jabr
w'al-muqabalah

= Ludus algebrae
et almucrabalaeque

Algorithmi de numero Indorum
'Dixit Algorismi'



Solving Quadratic Equations

Six types: (a, b, c all positive)

$$ax^2 = bx, ax^2 = b, ax = b,$$

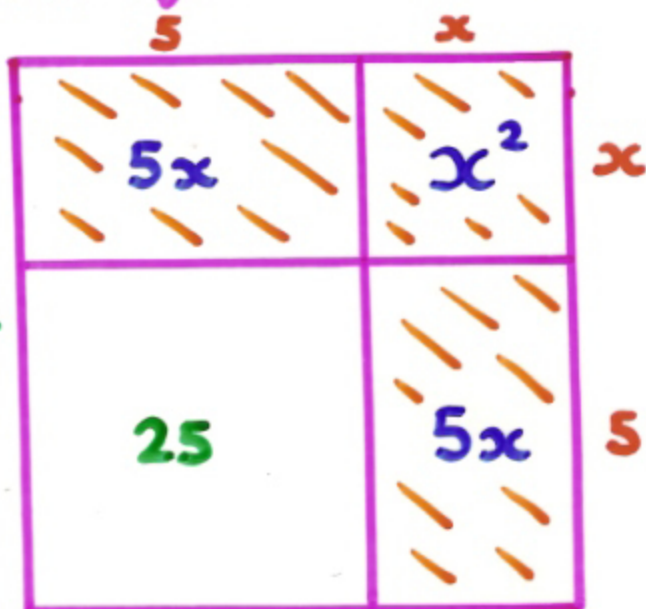
$$ax^2 + bx = c, ax^2 + c = bx, ax^2 = bx + c$$

'Roots and squares are equal to numbers'

$$x^2 + 10x = 39$$

$$(x+5)^2 = 39 + 25 = 64$$

$$\text{so } x+5 = 8, \underline{x=3}$$

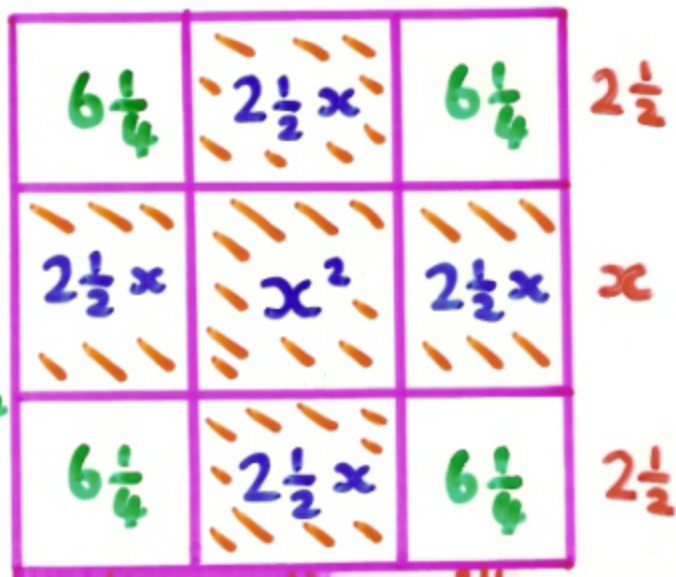


$$x^2 + 10x = 39$$

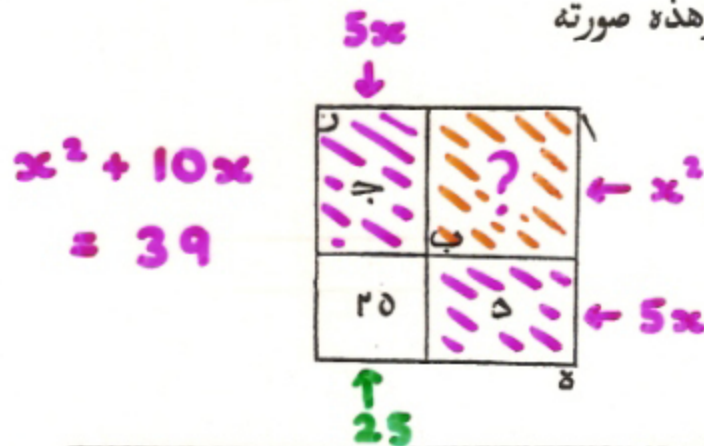
$$(x + 2\frac{1}{2} + 2\frac{1}{2})^2$$

$$= 39 + (4 \times 6\frac{1}{4}) = 64$$

$$\text{so } x+5 = 8, \underline{x=3}$$



علي تسعة وثلثين ليتم السطح الاعظم الذي هو سطح $\overline{ر ه}$ فبلغ ذلك كله اربعة وستين فاخذنا جذرها وهو ثمانية وهو احد اضلاع السطح الاعظم فاذا نقصنا منه مثل ما زدنا عليه وهو خمسة بقي ثلثة وهو ضلع سطح $\overline{أ ب}$ الذي هو المال وهو جذره والمال تسعة وهذه صورته



واما مال واحد وعشرون درهما يعدل عشرة اجذاره فانا نجعل المال سطحاً مربعاً مجهول الاضلاع وهو سطح $\overline{أ د}$ ثم نصم اليه سطحاً متوازي الاضلاع عرضه مثل احد اضلاع سطح $\overline{أ د}$ وهو ضلع $\overline{ه ن}$ والسطح $\overline{ه ب}$ فصار طول السطحين جميعاً ضلع $\overline{ج ه}$ وقد علمنا ان طوله عشرة من العدد لان كل سطح مربع متساوي الاضلاع والزوايا فان احد اضلاعه مضروباً في واحد جذر ذلك السطح وفي اثنين جذراه فلما قال مال واحد وعشرون يعدل عشرة اجذاره علمنا ان طول ضلع $\overline{ج ه}$ عشرة اعداد لان ضلع $\overline{ج د}$ جذر المال فقسمنا ضلع $\overline{ج ه}$ بنصفين علي نقطة

al-Khwarizmi's 'Algebra'

Omar Khayyam (c. 1100)

Equations in general:

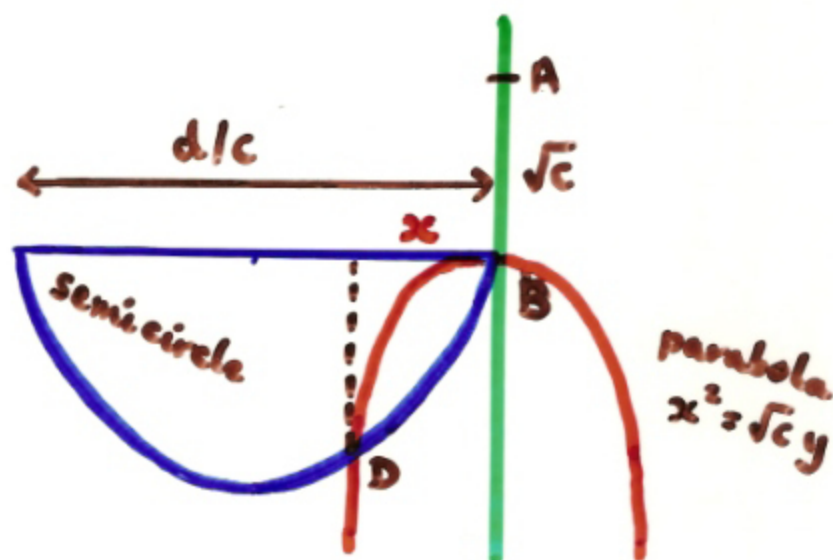
roots, squares, cubes,

square squares, square cubes, ...

A solid cube plus squares plus edges
equal to a number ($x^3 + ax^2 + bx = c$)

Also: $x^3 + bx = c$, $x^3 = bx^2 + d$, ... [14 in all]

$$x^3 + cx = d$$



Fibonacci (Leonardo of Pisa)



Liber abaci (1202)

Book of squares

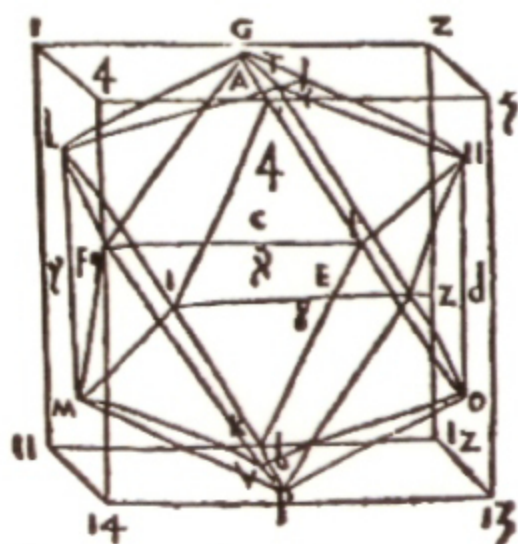
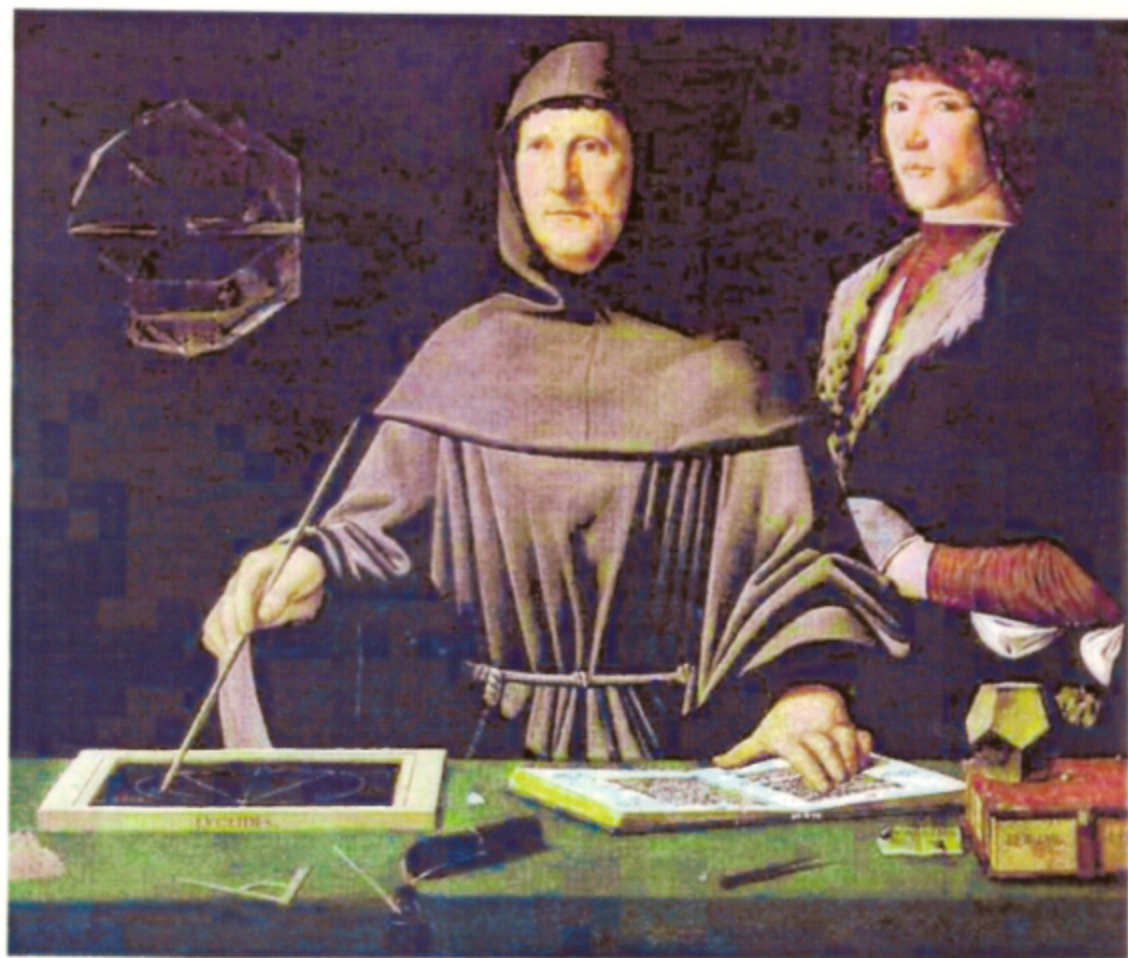
Hindu-Arabic numerals



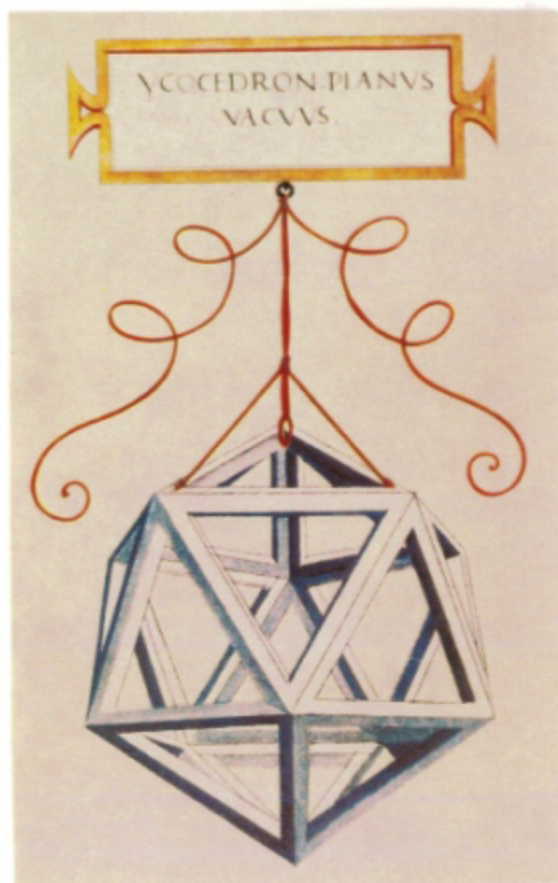
'Liber Abaci' - three problems

- There is a tree, $\frac{1}{4}$ and $\frac{1}{3}$ of which lie below ground; 21 palmi. How tall is the tree?
Guess 12, giving $3 + 4 = 7$ (instead of 21):
Scale up to give $3 \times 12 = 36$.
- If I buy 3 sparrows for a penny, turtle-doves 2 for a penny, and doves for two pence - and spend 30 pence for 30 birds, how many of each kind do I buy?
- If a lion eats a sheep in 4 hours, a leopard eats it in 5 hours, and a bear eats it in 6 hours, how long would they all take together?

Luca Pacioli



icosahedron
in a cube





IN MEMORY OF
ROBERT RECORDE,
THE EMINENT MATHEMATICIAN,
WHO WAS BORN AT TENBY, CIRCA 1510.
TO HIS GENIUS WE OWE THE EARLIEST
IMPORTANT ENGLISH TREATISES ON
ALGEBRA, ARITHMETIC, ASTRONOMY, AND GEOMETRY;
HE ALSO INVENTED THE SIGN OF
EQUALITY = NOW UNIVERSALLY ADOPTED
BY THE CIVILIZED WORLD.

ROBERT RECORDE
WAS COURT PHYSICIAN TO
KING EDWARD VI. AND QUEEN MARY.
HE DIED IN LONDON,
1558.

The whetstone of witte,

whiche is the seconde parte of
Arithmetike: containyng the extrac-
tion of Rvootes: The Cossike pynall
with the rule of Equation: and
the woorkes of Surde
Numbers.

Though many stones doe beare greate price,
The whetstone is for exercise
As needefull, and in woorkes as strange:
Dulle thinges and harde it will so chaunge,
And make them sharpe, to right good vse:
All artesmen knowe, thei can not chuse,
But vse his helpe: yet as men see,
Nee sharpenesse semeth in it to bee.

The grounde of artes did brede this stone:
His vse is greate, and more then one.
Here if you list your wittes to whette,
More sharpenesse thereby shall you gette.
Dulle wittes hereby doe greatly mende,
Sharpe wittes are fined to their fulle ende.
Not to proue, and praisse, as you doe finde;
And to your self be not vnkinde.

¶ These Bookes are to bee solde, at
the Weste doore of Poules,
by Iohn Byngstone.

Whetstone of Witte (1557)

The Arte

as their workes doe extende) to distinge it onely into two partes. Whereof the firste is, when one number is equalle vnto one other. And the seconde is, when one number is compared as equalle vnto .2. other numbers.

Alwaies willing you to remember, that you reduce your numbers, to their leaste denominations, and smalleste formes, before you procede any farther.

And again, if your equation be soche, that the greatestte denomination *Coslike*, be ioined to any parte of a compounde number, you shall tourne it so, that the number of the greatestte signe alone, maie stande as equalle to the reste.

And this is all that needeth to be taughte, concerning this worke.

Howbeit, for easie alteration of equations. I will propounde a fewe examples, because the extraction of their rootes, maie the more aptly bee wroughte. And to auoide the tedious repetition of these wordes: is equalle to: I will sette as I doe often in worke vse, a paire of paralleles, or Gemowe lines of one lengthe, thus: ———, because noe. 2. thynges, can be moare equalle. And now marke these numbers.

1. $14ze. + 15g. = 71g.$
2. $20ze. = 18g. = 102g.$
3. $26z. + 10ze = 9z. + 10ze + 213g.$
4. $19ze. + 192g. = 10z. + 108g. + 19ze$
5. $18ze. + 24g. = 8z. + 2ze.$
6. $34z. = 12ze = 40ze. + 480g. + 9z.$
1. In the firste there appeareth. 2. numbers, that is
14ze.

Cubic equations

A cube + squares + edges = a number

$$x^3 + ax^2 + bx = c$$

A cube + things = numbers

$$x^3 + cx = d$$

Cubes + squares = numbers

$$ax^3 + bx^2 = d$$

Scipione del Ferro }
Antonio Fiore } $x^3 + cx = d$

Niccolo Tartaglia } $x^3 + bx^2 = d$

Gerolamo Cardano

Rafael Bombelli

Fior's challenge to Tartaglia (1535)

1. Find me a number such that when its cube root is added to it, the result is 6.

...

$$[x^3 + x = 6]$$

15. A man sells a sapphire for 500 ducats, making a profit of the cube root of his capital.

How much is the profit?

...

$$[x^3 + x = 500]$$

17. A tree, 12 *braccia* high, was broken into two parts at such a point that the height of the part that was left standing was the cube root of the length of the part that was cut away. What was the height of the part left standing?

...

$$[x^3 + x = 12]$$

30. There are two bodies of 20 triangular faces whose corporeal areas together make 700 *braccia* and the area of the smaller is the cube root of the larger. What is the smaller area?

$$[x^3 + x = 700]$$

Q V E S I T I, ET INVENTIONI DIVERSE

DE NICOLO TARTAGLIA,

Di nouo restampati con vna Gionta al sesto libro, nella quale si
mostra duoi modi di redur vna Città inespugnabile.

*La diuisione, & continentia di tutta l'opra nel seguente foglio si
trouarà notata.*



... to enable me to remember the method
in any unforeseen circumstance,
I have arranged it as a verse in rhyme ...

*When the cube and the thing together
Are equal to some discrete number,*

$$x^3 + cx = d$$

Find two other numbers differing in this one.

$$u - v = d$$

*Then you will keep this as a habit
That their product shall always be equal
Exactly to the cube of a third of the things.*

$$uv = (c/3)^3$$

*The remainder then as a general rule
Of their cube roots subtracted
Will be equal to this principal thing.*

$$x = \sqrt[3]{u} - \sqrt[3]{v}$$

Solving a cubic equation

$$x^3 + 6x = 20$$

Find u and v so that

$$u - v = 20 \quad \text{and} \quad uv = (6/3)^3 = 8.$$

Since $v = u - 20$, we have

$$uv = u(u - 20) = u^2 - 20u = 8.$$

Solving this quadratic equation:

$$u = \sqrt{108} + 10.$$

$$\text{So} \quad v = u - 20 = \sqrt{108} - 10.$$

So

$$x = \sqrt[3]{u} - \sqrt[3]{v}$$

$$= \sqrt[3]{(\sqrt{108} + 10)} - \sqrt[3]{(\sqrt{108} - 10)}$$

$$= 2.$$

HIERONYMI CAR
 DANI, PRÆSTANTISSIMI MATHE
 MATICI, PHILOSOPHI, AC MEDICI,
 ARTIS MAGNAE.
 SIVE DE REGVLIS ALGEBRAICIS.
 Lib. unus. Qui & totius operis de Arithmetica, quod
 OPVS PERFECTVM
 inscripsit, est in ordine Decimus.



HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cosa uocant) nouis adinventionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta eruerint. Neque solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni equales fuerint, nodum explicant. Hunc autem librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmetice thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarètur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.

Cardano's Oath

To Tartaglia:

I swear to you, by God's holy Gospels, and as a true man of honour, not only never to publish your discoveries, if you teach me them, but I also promise you, and I pledge my faith as a true Christian, to note them down in code, so that after my death, no-one will be able to understand them.

Solving a quartic equation

$$\underline{x^4 = px^2 + qx + r}$$

$$(x^2 + y)^2 = (p + \underline{2y})\underline{x^2} + qx + (\underline{r + y^2})$$

For RHS a square we need:

$$(p + 2y)(r + y^2) = q^2/4 \quad [b^2 = 4ac]$$

$$(*) \quad 2y^3 + py^2 + 2ry + (pr - q^2/4) = 0$$

(*) then reduces to two quadratics.

e.g. $\underline{x^4 = x + 2}$

$$\begin{aligned} x = & \sqrt[3]{\sqrt{\frac{2075}{442368}} + \frac{1}{128}} - \sqrt[3]{\sqrt{\frac{2075}{442368}} - \frac{1}{128}} \\ & + \sqrt[3]{\sqrt{\frac{1051}{3456}} + \sqrt{\frac{2075}{442368}}} + \sqrt[3]{\sqrt{\frac{1051}{3456}} - \sqrt{\frac{2075}{442368}}} \dots \\ & + \frac{2}{3} - \sqrt[3]{\sqrt{\frac{2075}{442368}} + \frac{1}{128}} - \sqrt[3]{\sqrt{\frac{2075}{442368}} - \frac{1}{128}} \end{aligned}$$

Cardano's problem

Divide 10 into two parts
whose product is 40.

If the parts are x and $10-x$,
then $x(10-x) = 40$.

Cardano: $x = 5 + \sqrt{-15}$ or $5 - \sqrt{-15}$

'Nevertheless we will operate, putting
aside the mental tortures involved.'

$$\begin{aligned}x(10-x) &= (5 + \sqrt{-15})(5 - \sqrt{-15}) \\&= 25 - (-15) = 40.\end{aligned}$$

L'ALGEBRA OPERA

Di RAFAEL BOMBELLI da Bologna
Divisa in tre Libri.

*Con la quale ciascuno da se potrà venire in perfetta
cognitione della teorica dell'Arimetica.*

Con vna Tauola copiosa delle materie, che
in essa si contengono.

*Posta hora in luce à beneficio delli Studiosi di
detta professione.*



IN BOLOGNA,
Per Giovanni Rossi. MDLXXIX.
Con licenza de' Superiori

Bombelli and complex numbers

$$x^3 = 15x + 4$$

Solutions: $4, -2+\sqrt{3}, -2-\sqrt{3}$

Cardano's method yields

$$x = \sqrt[3]{(2+\sqrt{-121})} - \sqrt[3]{(-2+\sqrt{-121})}$$

first appearance of complex numbers

Bombelli calculated that:

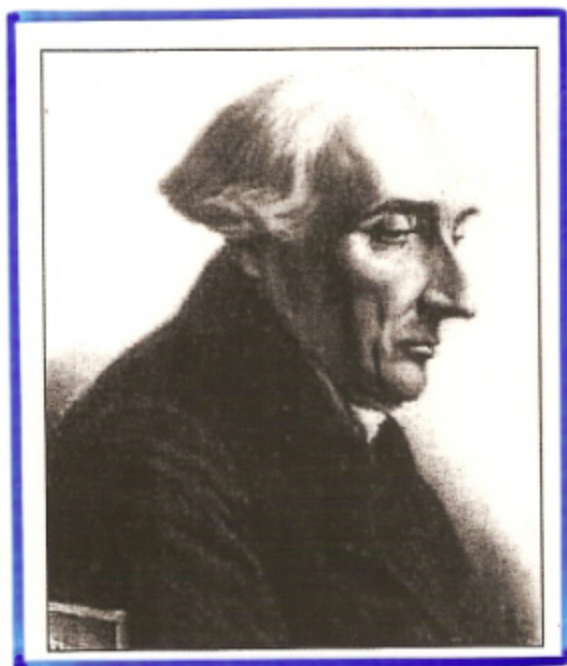
$$(2+\sqrt{-1})^3 = 2+\sqrt{-121},$$

$$(2-\sqrt{-1})^3 = 2-\sqrt{-121}.$$

$$\text{So } x = (2+\sqrt{-1}) - (-2+\sqrt{-1}) = 4.$$

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Lagrange's reduction methods



$$x^3 + nx + p = 0$$

$$\text{put } x = y - \frac{n}{3y}$$

$$y^6 + py^3 - \frac{1}{27}n^3 = 0$$

quadratic in y^3

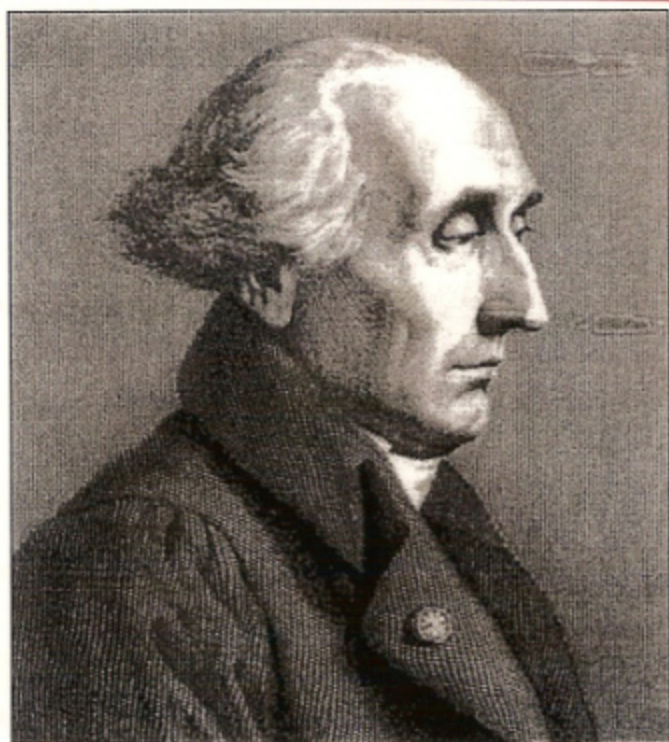
six $\underline{y}$ s $\rightarrow$ three $\underline{x}$ s

Quartic: $x^4 + nx^2 + px + q = 0$

can be reduced to

$$y^3 - \frac{n}{2}y^2 - qy + \frac{4nq - p^2}{8} = 0$$

cubic in y , three $\underline{y}$ s $\rightarrow$ twelve $\underline{x}$ s
 $\rightarrow$ four different $\underline{x}$ s



Permuting the solutions

$$x^3 + ax^2 + bx + c = 0$$

solutions p, q, r

$$x^3 + ax^2 + bx + c = (x-p)(x-q)(x-r)$$

$$c = -pqr, \quad b = pq + pr + qr,$$

$$a = -(p+q+r)$$

Permute the solutions – no change

$$pq + 2r : \quad pr + 2q \quad qr + 2p \quad (3 \text{ values})$$

$$p + 2q + 5r : \quad p + 2r + 5q, \dots \quad (6 \text{ values})$$

...

(always divides 6)

$$(p-q)(q-r)(r-p) \quad (2 \text{ values})$$



Paolo Ruffini

(1765 - 1822)

The algebraic solution
of equations of degree
greater than 4 is always
impossible.

Behold a very important theorem
which I believe I am able to assert ...
to present the proof of it is the main
reason for publishing this volume ...



Niels Henrik Abel

(1802-1829)

proved impossibility
of solving quintic equations

Evariste Galois

(1811-1832)

developed criteria for
deciding which equations
can be solved.



Galois to Chevalier, 29 May 1832

My dear friend,

I have done several new things in analysis.
Some concern the theory of equations;
others, integral functions.

In the theory of equations I have found
out in which cases the equations are
solvable by radicals, which has given me
the occasion to deepen the theory and to
describe all the transformations admitted
by an equation, even when it is not
solvable by radicals.

...

You will publicly beg Jacobi or Gauss to
give their opinion not of the truth but of the
importance of the theorems.

After this, there will, I hope, be people
who will find it to their advantage to
decipher all this mess.

Four bells: Plain Bob Minimus

e	1	2	3	4
o	<u>2</u>	<u>1</u>	<u>4</u>	<u>3</u>
e	2	<u>4</u>	<u>1</u>	3
o	<u>4</u>	<u>2</u>	<u>3</u>	<u>1</u>
e	4	<u>3</u>	<u>2</u>	1
o	<u>3</u>	<u>4</u>	<u>1</u>	<u>2</u>
e	3	<u>1</u>	<u>4</u>	2
o	<u>1</u>	<u>3</u>	<u>2</u>	<u>4</u>
e	1	3	<u>4</u>	<u>2</u>
o	<u>3</u>	<u>1</u>	<u>2</u>	<u>4</u>
e	3	<u>2</u>	<u>1</u>	4
o	<u>2</u>	<u>3</u>	<u>4</u>	<u>1</u>
e	2	<u>4</u>	<u>3</u>	1
o	<u>4</u>	<u>2</u>	<u>1</u>	<u>3</u>
e	4	<u>1</u>	<u>2</u>	3
o	<u>1</u>	<u>4</u>	<u>3</u>	<u>2</u>
e	1	4	<u>2</u>	<u>3</u>
o	<u>4</u>	<u>1</u>	<u>3</u>	<u>2</u>
e	4	<u>3</u>	<u>1</u>	2
o	<u>3</u>	<u>4</u>	<u>2</u>	<u>1</u>
e	3	<u>2</u>	<u>4</u>	1
o	<u>2</u>	<u>3</u>	<u>1</u>	<u>4</u>
e	2	<u>1</u>	<u>3</u>	4
o	<u>1</u>	<u>2</u>	<u>4</u>	<u>3</u>
e	1	2	<u>3</u>	<u>4</u>

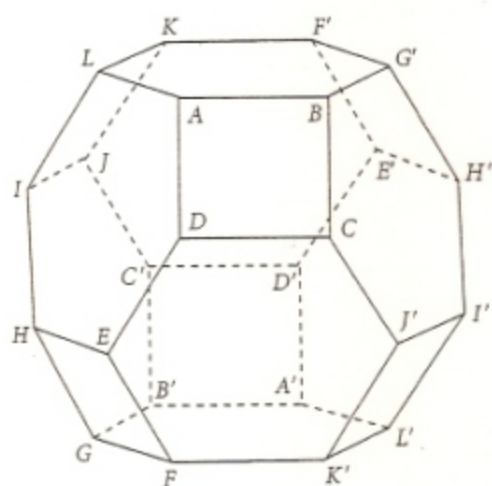
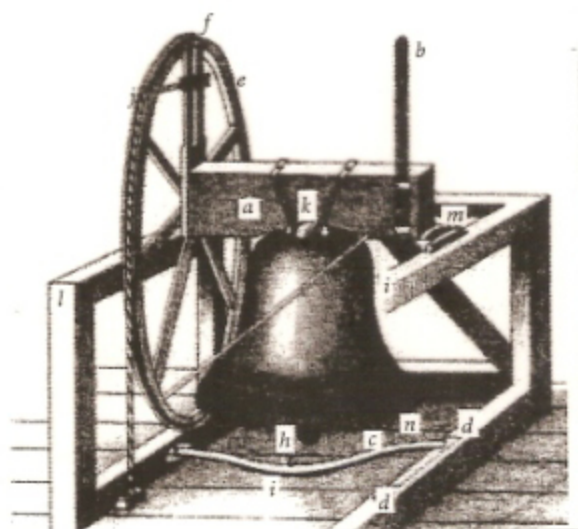
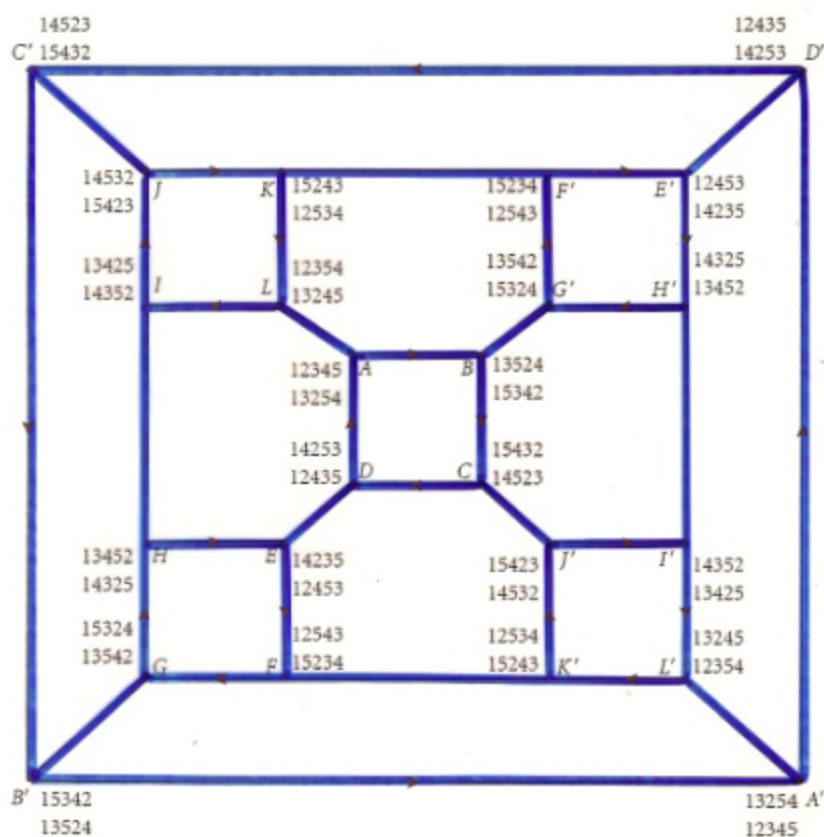
Interchange
consecutive bells :

1	2	3	4
×		×	
2	1	4	3
	×		
2	4	1	3
...			

(even)

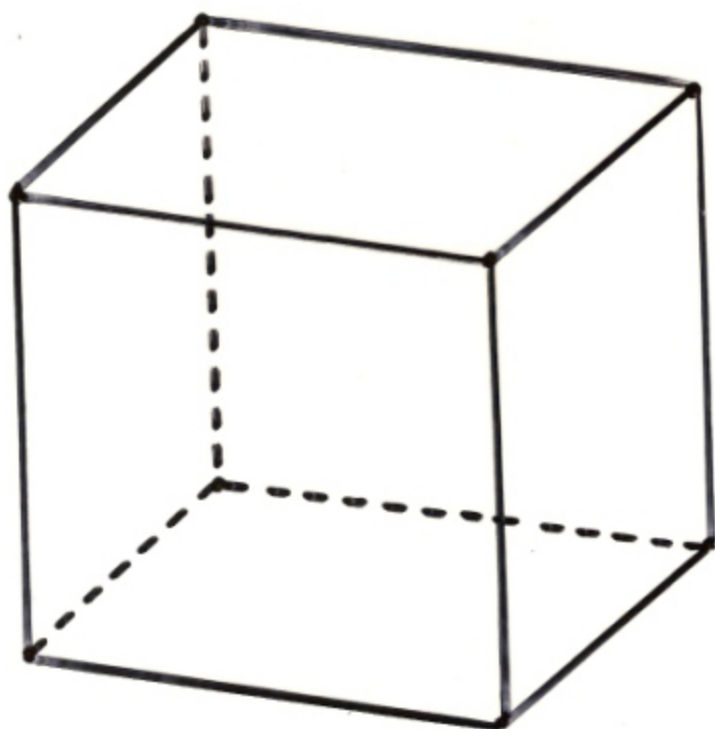
(odd)

Five bells : Plain Bob Doubles



truncated
octahedron

Symmetries of a Cube

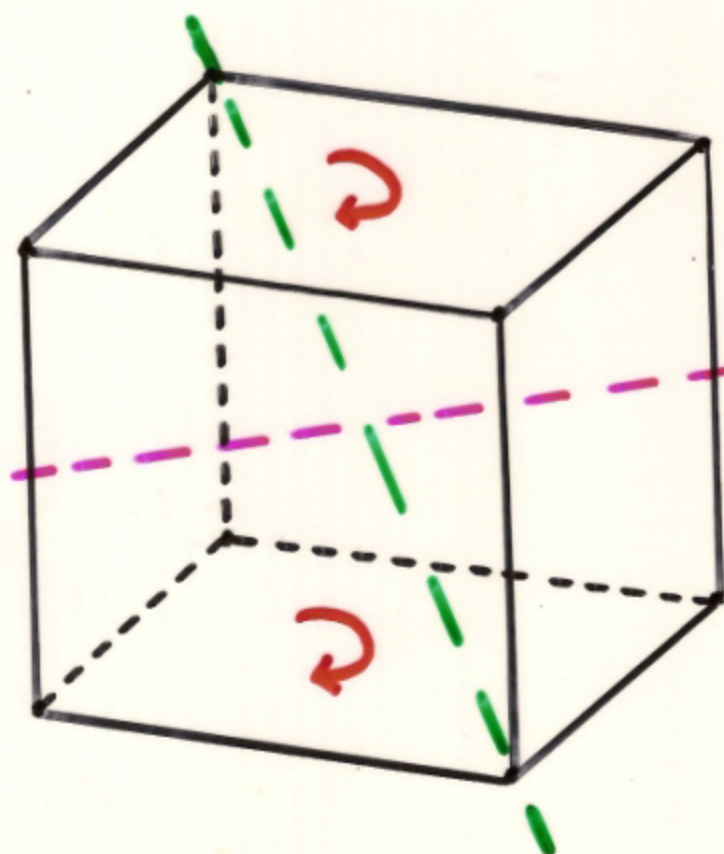


24 rotations

24 'indirect' symmetries

(reflection + rotation)

Symmetries of a Cube



Rotations

1 identity

9 face rotations

8 vertex rotations

6 edge rotations

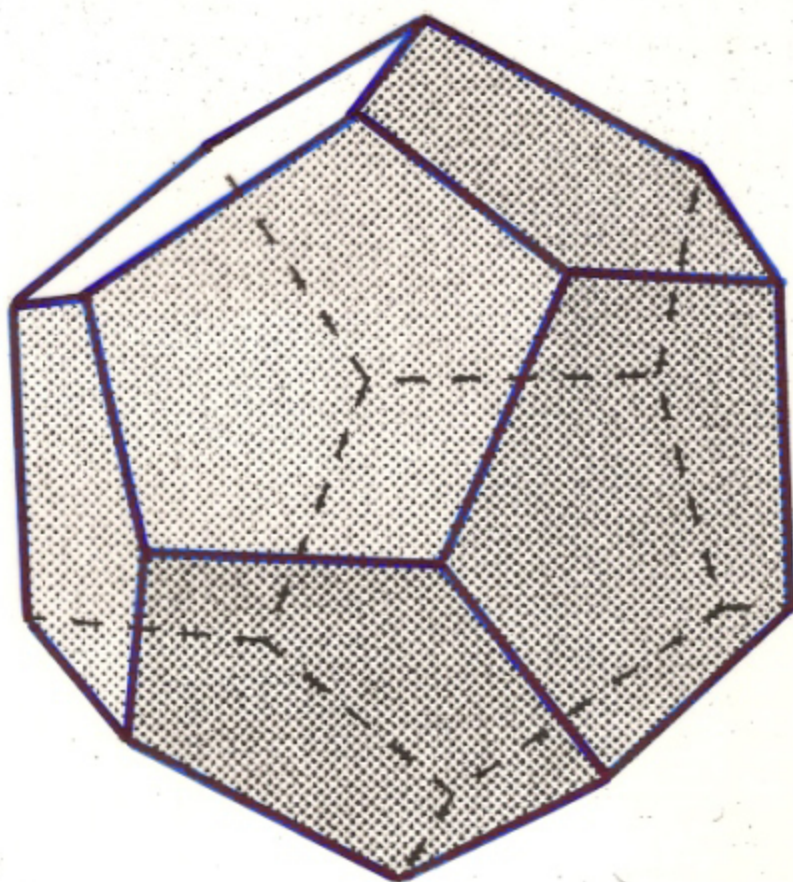
Total: 24

24 rotations

24 'indirect' symmetries

(reflection + rotation)

Rotations of a Dodecahedron



60 rotations

Abstract Groups

Take a set G of elements,
and a rule for combining them $(*)$.

For a group, we have:

(1) Closure: if a and b are in G ,
then so is $a * b$.

(2) Associativity: if a, b, c are in G ,
then $(a * b) * c = a * (b * c)$

(3) Identity: there is an element e
such that $a * e = e * a = a$ for all a .

(4) Inverses: for each a in G , there is
an inverse a^{-1} such that $a * a^{-1} = e$.

[(5) Abelian: if a and b are in G ,
then $a * b = b * a$.]

Remembering the group axioms

Closure

Associative

Inverses

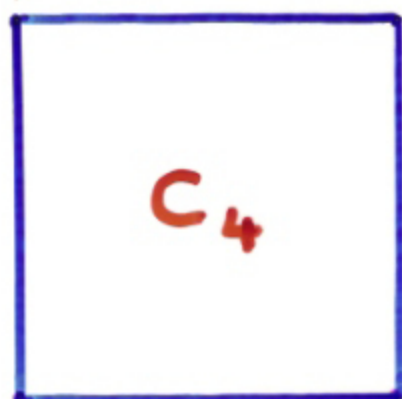
Neutral element

not necessarily ABELian

Examples :

- permutations of 4 bells
- symmetries of a cube
- rotations of a dodecahedron
- addition of integers
- multiplying positive real numbers

Cyclic Groups

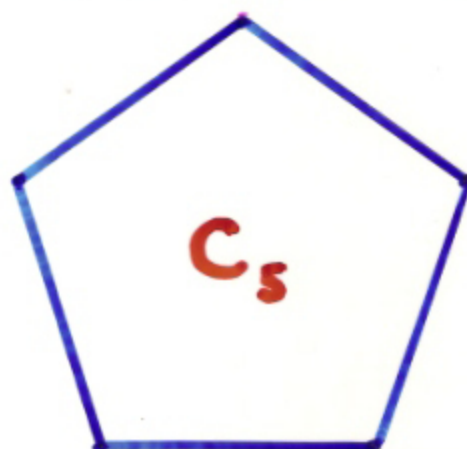


rotations of a square through 0, 1, 2, 3 right angles:

$1 + 3 = 0, 3 + 2 = 1, \dots$
(arithmetic 'mod 4')

rotations of a pentagon through 0, 1, 2, 3, 4 turns:

$2 + 3 = 0, 3 + 4 = 2, \dots$



Classification theorem: Every abelian group is obtained by combining cyclic groups.

Simple groups

- 'building blocks' for groups in general
- groups that 'contain no other groups inside them' – they have no proper normal subgroups
- Cyclic groups with a prime number of elements: $C_5, C_7, C_{101}, \dots$
- rotations of a dodecahedron (A_5)
- A_n ($n \geq 5$): even permutations of $\{1, 2, \dots, n\}$
- groups of 'Lie type' (related to matrices)
- 26 'sporadic groups'

[Monster group: rotations in 196883-dimensional space]