G R E S H A M

C O L L E G E

FOUR CENTURIES OF LOGARITHMS

A Lecture by

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GRESHAM COLLEGE

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- to foster academic consideration of contemporary problems;

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Gresham Lecture
Four Centuries of Logarithms

Ian Stewart 24.1.97

The first Gresham Professor of Geometry, appointed four hundred years ago, was Henry Briggs. Among his claims to fame is the invention of *logarithms* — in the form that we now use. Logarithms themselves first came to light in the work of John Napier, and the idea seems to have originated around 1594. Briggs made a significant improvement on Napier's original conception, making the idea far more practical (and mathematically more natural).

For most of the intervening four centuries the main use of logarithms was to facilitate arithmetical calculations. Logarithms reduce multiplication to addition, which is easier and much quicker. They also form the basis of the slide rule. Within the last twenty years, however, slide rules have been replaced by calculators, and logarithms are seldom, if ever, used for artihmetical purposes.

So have logarithms gone the way of the dodo?

Let's see how they have fared over the centuries.

Century 1: 1594 to 1699

Napier published his system in 1614, but he wrote that it took him some twenty years to invent it. In modern notation, the underlying idea is the power law

$$x^a x^b = x^{a+b}.$$

To multiply two numbers u and v, find a and b so that

$$u = x^a$$
$$v = x^b$$

and observe that uv = w where w is such that

$$w = x^{a+b}.$$

Again in modern notation, we have

$$a = \log_x u$$
$$b = \log_x v$$
$$a+b = \log_x w.$$

However, this is not exactly how Napier proceeded. In modern notation, his method led to the value

$$\text{Nap log } x = 10^{10}[\log_e(10^{10}) - \log_e x]$$

where e = 2.71828... is the 'base of natural logarithms'. But modern notation, and even the concept of e, did not exist at that time.

Henry Briggs was the first Gresham Geometry Professor and also the first Savilain Professor of Geometry at Oxford. In 1615 he visited Napier at his home in Scotland, and they discussed ways to improve the concept of a logarithm to make it more practical to use. Briggs in effect proposed using powers of 10, so that when

$$10^a = y$$

then

$$a = \log_{10} y.$$

One major advantage is that

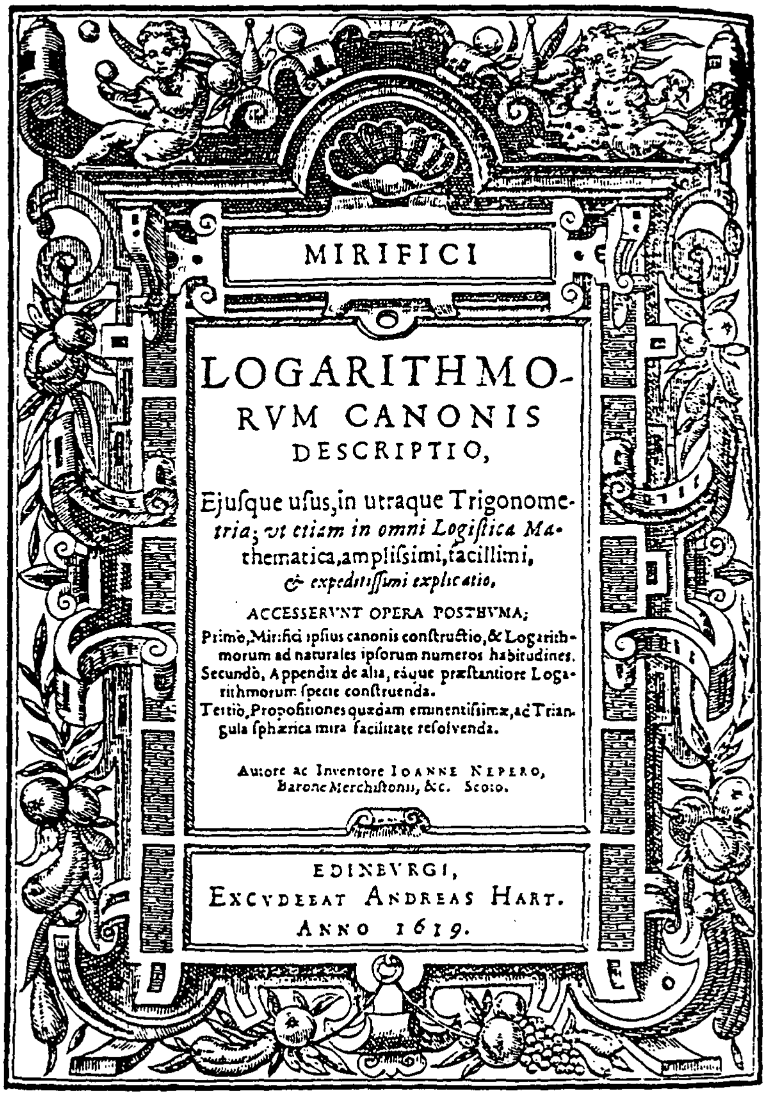
$$\log_{10}(10y) = 1 + \log_{10} y$$

$$\log_{10}(100y) = 2 + \log_{10} y$$

$$\log_{10}(1000y) = 3 + \log_{10} y$$

and so on.

Briggs undertook to calculate a table of logairthms to base 10, and to publish it. It appeared as *Arithmetica Logarithmica* in 1624.



MIRIFICI

LOGARITHMORVM CANONIS DESCRIPTIO,

Ejusque usus, in utraque Trigonometria; *ut etiam in omni Logistica Mathematica*, amplissimi, facillimi, *& expeditissimi explicatio*.

ACCESSERVNT OPERA POSTHVMA;

Primò, Mirifici ipsius canonis constructio, & Logarithmorum ad naturales ipsorum numeros habitudines.
Secundò, Appendix de alia, eaque præstantiore Logarithmorum specie construenda.
Tertiò, Propositiones quædam eminentissimæ, ad Triangula sphærica mira facilitate resolvenda.

Autore ac Inventore IOANNE NEPERO, Barone Merchistonii, &c. Scoto.

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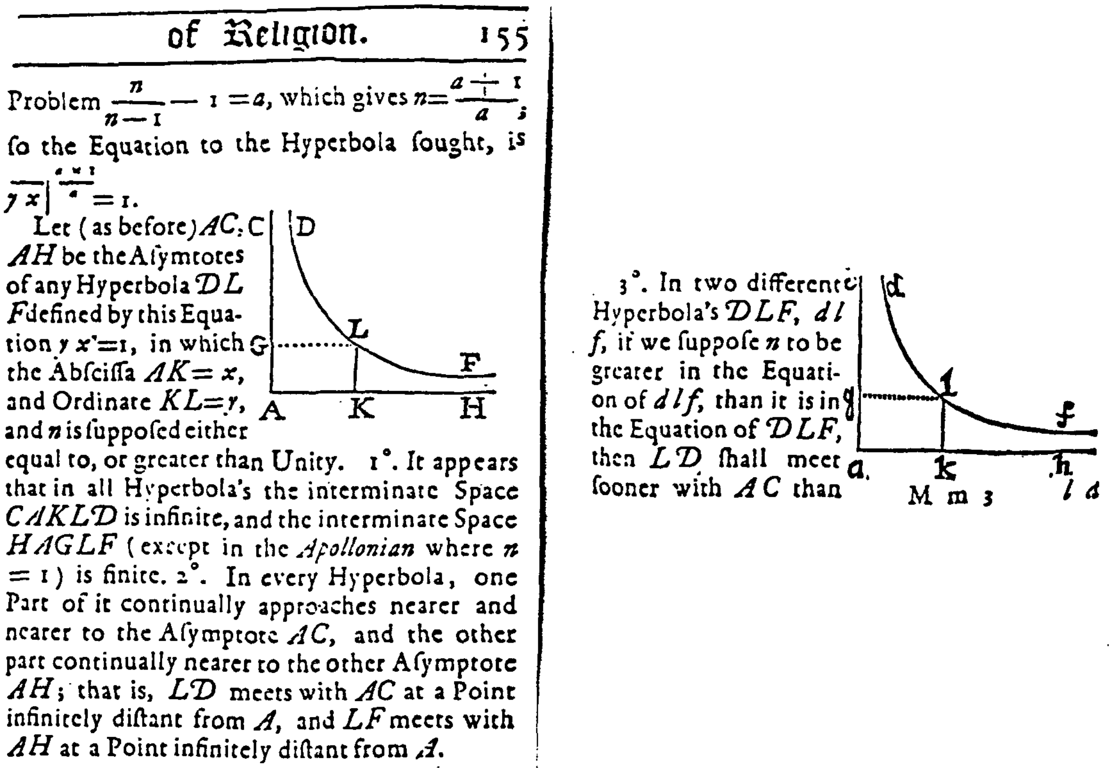
Fig.1 Title page of Napier's Logarithm Tables.

Later in the same century Newton introduced the calculus, and it was discovered that the logarithm was related to the area under a hyperbola.



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Problem $\frac{n}{n-1} - 1 = a$, which gives $n = \frac{a+1}{a}$; ſo the Equation to the Hyperbola ſought, is $\overline{yx}|^{\frac{a+1}{a}} = 1$.

Let (as before) *AC*, *AH* be the Aſymtotes of any Hyperbola *DLF* defined by this Equation $yx^n = 1$, in which the Abſciſſa *AK* = *x*, and Ordinate *KL* = *y*, and *n* is ſuppoſed either equal to, or greater than Unity. 1°. It appears that in all Hyperbola's the interminate Space *CAKLD* is infinite, and the interminate Space *HAGLF* (except in the *Apollonian* where *n* = 1) is finite. 2°. In every Hyperbola, one Part of it continually approaches nearer and nearer to the Aſymptote *AC*, and the other part continually nearer to the other Aſymptote *AH*; that is, *LD* meets with *AC* at a Point infinitely diſtant from *A*, and *LF* meets with *AH* at a Point infinitely diſtant from *A*.

3°. In two different Hyperbola's *DLF*, *dlf*, if we ſuppoſe *n* to be greater in the Equation of *dlf*, than it is in the Equation of *DLF*, then *LD* ſhall meet ſooner with *AC* than

Fig.2 A page from George Cheyene's *Philosophical Principles of Religion* discussing the quadrature of the hyperbola.

Century 2: 1700 to 1799

This is the century of Leonhard Euler, who made the logarithm (and its inverse function, the exponential) the basis of analysis.

Euler introduced the symbol e for the base of natural logarithms, defining it as

$$e = \lim_{n \to \infty} (1 + 1/n)^n.$$

Then, if $y = e^x$, we have $x = \log_e y$ (or just $\log y$).

He also showed that (using complex numbers in which $i = \sqrt{-1}$)

$$e^{ix} = \cos x + i \sin x$$

linking logarithms and exponentials to trigonometry.

Century 3: 1800 to 1899

This century witnessed the flourishing of *complex* analysis, and there was considerable controversy over the value of

$$\log(-1).$$

Eventually Euler sorted it all out by arguing that

$$\log(-1) = i\pi + 2k\pi$$

for any integer k. Complex logarithms are *many-valued.*

Gauss used this fact to prove the 'fundamental theorem of algebra' that any polynomial equation of degree d over the complex numbers has d complex solutions.

Cauchy used it as the basis for a method of studying complex analytic functions.

Century 4: 1900 to 1997

In the modern era, the roles of the logarithm and the exponential have become inordinately varied. Here are just three areas:

Dynamical Systems

The solutions of systems of linear differential equations

$$dx/dt = Ax,$$

where x is a vector and A a matrix, are given by the exponential function:

$$x(t) = x_0 e^{At}.$$

Fractals

The fractal dimension of a self-similar fractal is given by a formula involving logarithms. For example the Cantor set has dimension $\log 2 / \log 3 = 0.6309$.

Probability

Benford's Law, a probabilistic curiosity used, among other things, by tax authorities to detect fraud, holds that in any collection of natural data the probability that a given number has the first digit n is $\log(n+1)-\log(n)$. So 1 is more likely than 2, and so on. Examples include the sizes of islands in the Bahamas and currency rates in newspapers.

<u>*Logarithms are alive and well.*</u>

FURTHER READING

Eli Maior, *e: The story of a number*, Princeton University Press 1994.

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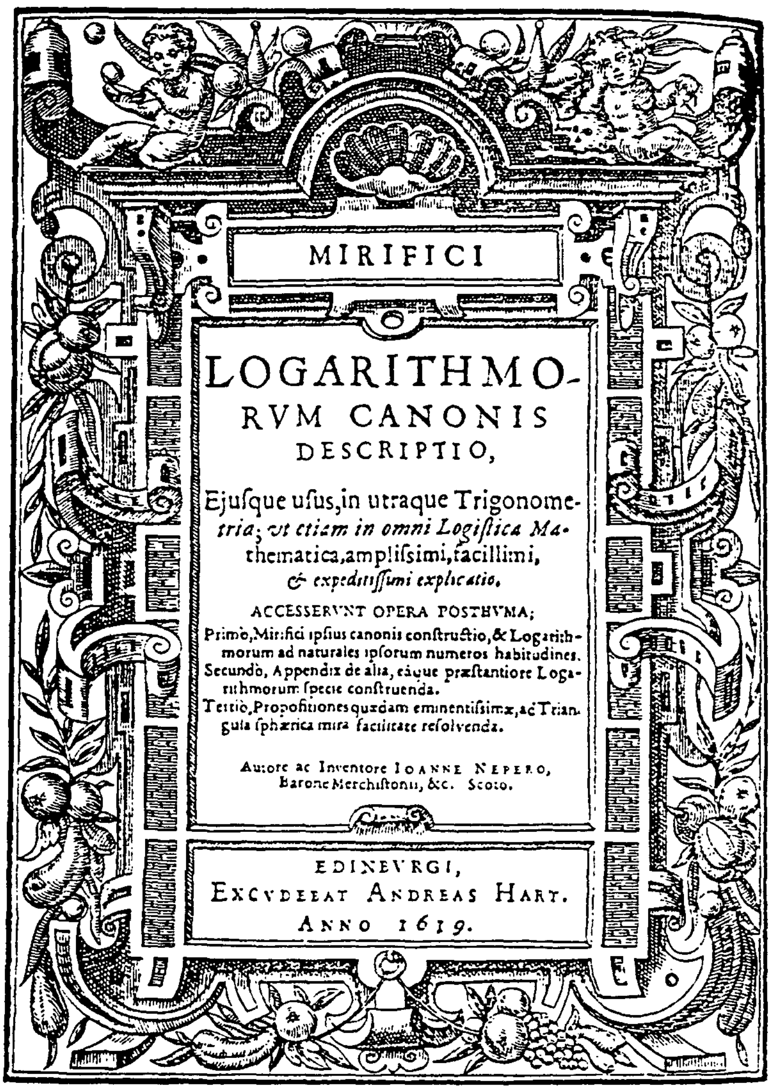
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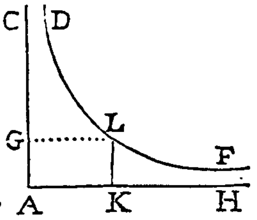

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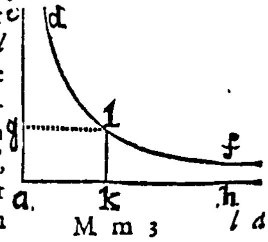

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