Mathematical Journeys into Fictional Worlds
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9 February 2021

In this lecture we explore some of the ways that mathematical ideas appear in fiction. We'll go with Gulliver to Lilliput, meet the giant spiders of Hogwarts, chat with the King of Line-land, and even drop in on Captain Ahab.

Introduction

Travel to different realms, both real and fictitious, can take us out of ourselves and allow us to see the peculiarities and prejudices of our own world more clearly. In Gulliver's Travels, the literal littleness of the Lilliputians is reflected in the pettiness of their concerns – starting a war over the best way to break an egg, for example. Their arrogance satirises our own – mired in political squabbles and not seeing the bigger picture. Gulliver travels to many lands, and while of course we do not judge a satirical novel by the plausibility of its science, it is still entertaining to ask what life would be like for the Lilliputians and Brobdingnagians. Mathematics can tell us a great deal about this. We can also apply it to the wealth of other examples of large and small people and creatures in books, films and on TV. Given that both Jonathan Swift and Voltaire (who wrote about a giant named Micromégas in a satirical story) explicitly mention calculations made by mathematicians, I think it is fair game to look at whether those calculations hold water. Once we do that, we are led down many avenues of mathematical exploration, some of which we will follow in the lecture. As we'll see, there are several mathematical principles that we can use to shed some light on these questions. These principles also have many real-world implications.

Our next starting point in the lecture is another satire, Edwin A. Abbot's *Flatland.* Its protagonist, A. Square, inhabits an entirely two-dimensional world, but through his visits to worlds in different dimensions learns much about the prejudices and limitations of his own world (many of which mirror the society of the Victorian Britain in which Abbot lived). The use of geometrical ideas is explicit throughout, in particular the concept of dimension, which we will discuss.

We'll finish with a brief look at some of the many ways in which authors have used mathematical metaphors in fiction, highlighting a few of my favourite examples.

There is another way mathematics can be used in fiction, and that is in the structure of the writing itself. This is the topic of the next lecture in the series, where we will discuss authors such as Georges Perec, Jorge Luis Borges and Eleanor Catton.

Giants and Lilliputians

In the 1726 novel *Gulliver's travels* by Jonathan Swift [1], Lemuel Gulliver visits many strange lands. He first visits Lilliput, where the people are tiny, and then Brobdingnag, where they are giants, twelve times our height. These are far from the only giants in popular culture. To ascertain if such creatures can exist, we have to know the claimed dimensions of the giant in question. Fairy tales don't tend to give exact dimensions, but we do have information in other cases. King Og of Bashan (whom Moses meets in the Book of Deuteronomy in the Bible) was sufficiently large that his bed was 13 feet long, so he was perhaps twice normal height. Rubeus Hagrid, gamekeeper at Hogwarts School of Witchcraft and Wizardry, is half-giant. He is described as twice the height of a normal man, 'and three times as wide'. Many films feature people growing and shrinking, with some of the silliest dating to the B-movie era of the 1950s and 1960s. One example is 1957's *The Amazing Colossal Man* (100ft tall), which was followed by *Attack of the 50ft woman* in 1958. On TV, there was *Land of the Giants* (originally airing from 1968 – 1970). It was set in the futuristic year of 1983, when explorers from Earth fell through a "space warp" to a giant version of Earth where everything was scaled up by a factor of 12.

Our explorations will be under the assumption that the laws of physics as we understand them apply. Omnipotent deities can perform whatever miracles they like; Albus Dumbledore, Headmaster of Hogwarts, can no doubt produce an enlargement charm; perhaps science-fiction giants' bones are made of unobtanium. So, if we do show that a particular creature 'cannot' exist, we could instead frame this as stating that if we do see such a creature, it is definitive proof of the existence of miracles, magic or some other way to break the laws of physics.

The Square-Cube Law

The single most important tool in our study of large and small things is the Square-Cube Law. It says that if an object is changed in size by a factor of k in all linear dimensions (length, width, height), then its volume changes by a factor of k^3, but its cross-sectional area and its surface area change by a factor of k^2.

Studies have shown that the human femur (thigh bone) will break under about 10 times the pressure it normally carries. Now, pressure $= \frac{\text{force}}{\text{area}}$, while, letting g be the acceleration due to gravity, the force on the bone is given by force $= mass \times g$. Meanwhile, assuming a fixed average density, the mass of a scaled human is proportional to their volume. Putting all this together, we get that the force on your bones is proportional to your volume, and hence to the cube of your height. Meanwhile, the cross-sectional area of bone is proportional to the square of your height. It's an example of the Square-Cube Law. Thus, pressure $\propto \frac{(\text{height})^3}{(\text{height})^2} =$ height. This puts a definite upper limit on the possible size of a giant, assuming his dimensions are like humans, but just scaled up.

The Brobdingnagians are twelve times the size of Gulliver. This means that the pressure on their bones is twelve times the pressure on ours. Their legs would break as soon as they tried to move! Similar considerations apply to Giant Pope and Giant Pagan of Bunyan's *Pilgrim's Progress*, and, sadly, to Roald Dahl's much-loved *BFG*. Hagrid is described as having twice the normal height and three times the normal width. This would mean his mass is about 18 times ours, and the cross-sectional area of his bones is about 9 times ours. Thus, the pressure on his bones is twice what it is on ours – this means he could still walk around but he would be very prone to getting broken bones, and definitely couldn't jump or run at any speed. The same goes for King Og of Bashan – he would be much weaker than Moses, and more liable to injury.

Micromégas

In Voltaire's short story [2], the eponymous Micromégas comes from a planet orbiting the star Sirius. He visits our solar system, and meets the people of Saturn, who are 6,000 feet tall, before travelling to earth, where he can only communicate with the atom-like humans there with an ear-trumpet he makes (for some reason) from his fingernail clippings. At 120,000 feet tall, Micromégas would collapse under his own weight on Earth, but would different gravitational forces allow such a megabeing to exist? Voltaire writes: *Certain geometers, always of use to the public, will immediately take up their pens, and will find that since Mr. Micromégas, inhabitant of the country of Sirius, is 24,000 paces tall, which is equivalent to 120,000 feet, and since we citizens of the earth are hardly five feet tall, and our sphere 9,000 leagues around; they will find, I say, that it is absolutely necessary that the sphere that produced him was 21,600,000 times greater in circumference than our little Earth. Nothing in nature is simpler or more orderly.* (Some editions have “algebraists” rather than “geometers”).

Now, if the calculation is just scaling up the circumference by a factor of 24,000, then it’s wrong, because $9{,}000 \times 24{,}000 = 216{,}000{,}000$. A bigger problem is that the acceleration due to gravity on the surface of a planet is proportional to its mass divided by the square of its radius. If the Sirian planet had the same density as ours, its mass would increase with the cube of the scaling factor, but we are only dividing by the square of the radius. Therefore, the Sirian planet would have 24,000 times the gravity that earth does. Perhaps the Sirian planet has a lower density. The least dense planets we know of are known as ‘super-puff’ planets. Their density is about 1% of ours, meaning a best-case scenario of 240 times our gravity. Micromégas is a satire on the contrast between how significant and important we think ourselves (very), and how significant and important we actually are (not at all). It is quite possible that Voltaire introduced the first error deliberately to show the fallibility even of such highly esteemed people as geometers! Alternatively, his arithmetic may just not have been very good.

Tiny People

Let’s meet some examples of miniature people.

- Here is Gulliver talking about Lilliput. *His Majesty’s Mathematicians, having taken the Height of my Body by the help of a Quadrant, and finding it to exceed theirs in the Proportion of Twelve to One, they concluded from the Similarity of their Bodies, that mine must contain at least 1,724 of theirs, and consequently would require as much Food as was necessary to support that number of Lilliputians.* Actually, that 1,724 isn’t quite right, as we’ll see later.
- Mary Norton’s popular series of children’s books feature a family of Borrowers – Arrietty, Pod and Homily. Borrowers live in human houses, under floorboards, in grandfather clocks, or behind the skirting boards, and ‘borrow’ things like needles, matchboxes and so on, out of which they make furniture and tools. They are an estimated one sixteenth of our size.
- A common B-movie trope is that of a shrink-ray, miniaturising potion or similar plot device that causes people and things to get smaller. The evil *Doctor Cyclops* (1940) uses a ‘radiation chamber’ to shrink his victims to a height of twelve inches (about a factor of six). *The Incredible Shrinking Man* (1957) sees the protagonist shrinking inexorably and irreversibly after encountering a mysterious ‘fog’. More recently we have *Honey I shrunk the Kids* (1989), where an inventor accidentally shrinks his kids to a height of a quarter of an inch – about a factor of 200, and *Downsizing* (2017), where Matt Damon shrinks to 5 inches tall.

What would life be like for these little people? We mostly focus on the Lilliputians, but similar observations apply for the other examples. See [3] for a discussion of what life is like for Borrowers.

Moving Around

Of course, our Lilliputians wouldn’t have the troubles that giants do with carrying their own weight. In fact, quite the reverse – they would be proportionately stronger. This means that some of the perilous situations that mini-people face in films are less perilous than we might think. In the charming Studio Ghibli Borrowers film *Arrietty* (2010), we see Pod abseiling down a table leg with a complicated bit of kit made from string and a safety pin. Meanwhile, in B-movie world a shrink-ray victim is trapped in a kitchen sink, unable to climb out.

Falling from height is dangerous because when we hit the ground the kinetic energy that has built up is released very suddenly. Smaller creatures have several advantages over us in this regard. When we fall, we accelerate under gravity; however, there is a small counteracting force in the other direction due to air resistance. Air resistance is proportional to the velocity and to the cross-sectional area of the falling body (which is why we use parachutes to increase our area and slow our fall). Terminal velocity is when these two forces cancel out, such that our net acceleration is zero. For humans, terminal velocity is roughly 50m/s. Now, the force of gravity is proportional to mass (it is Mg Newtons, where M is the mass in kg and g $\approx$ 9.81 is acceleration due to gravity), whereas resistance is proportional to area. Thus, if our dimensions change by a factor of k, the force due to gravity is multiplied by $k^3$and the resistance is multiplied by k^2 – another instance of the Square-Cube Law. Hence, terminal velocity is changed by a factor of k. The terminal velocity of a Lilliputian is therefore $\frac{1}{12} \times 50 = 4.2$m/s.

The kinetic energy of an object of mass M moving at velocity v is $\frac{1}{2}Mv^2$. The force required to stop that object over a small, fixed distance is proportional to this energy. Assuming as usual that our human-like creatures are just standard humans scaled equally in every linear dimension, the mass is proportional to height h cubed. That is, the force on impact at velocity v is proportional to h^3v^2. Meanwhile, the force of impact the body can withstand is proportional to the area impacting the ground, which is proportional to h^2. At maximum survivable velocity v, we have that $h^2 \propto h^3v^2$, and hence $v \propto \sqrt{\frac{1}{h}}$. Our Lilliputians are one twelfth of our height. Therefore, they can survive an impact at $\sqrt{12}$ times the velocity that we can survive. NASA has calculated 12m/s as an upper threshold for a survivable speed of impact – above this there is an increasing chance of death or severe injury. This implies that Lilliputians can survive about $12\sqrt{12} \approx 42$ m/s. Thus, given that their terminal velocity is 4.2 m/s, a Lilliputian can survive a fall from any height! By contrast, if we think about jumping, to jump a given height requires an amount of energy roughly proportional to your weight. The amount of energy muscles produce is also roughly proportional to their weight. Hence, the scaling factors cancel and in fact the height a scaled up or down human can jump to is almost independent of the scaling factor. As a consequence, the fact that a normal sized human can jump about a metre into the air implies that Lilliputians have this same jumping capacity. Thus, the miniature human in the B-movie trapped in a pint glass, or some such calamity, can simply jump out and escape quite easily. People commonly say that if a flea were the size of a human, it would be able to jump over a house. In fact, a giant flea could jump to about the same height as its normal-sized counterpart – about 18cm.

Eating

The Lilliputians calculate that as Gulliver's volume is 12^3 times theirs, he must need $12^3 = 1{,}728$ times as much food as them each day. (In fact, the original text says 1,724, with some editions since correcting this.) However, even the 1,728 number is not quite right; it misses an important fact about metabolic rates of creatures of different sizes. To stay alive, we must take in enough energy to keep all our organs going, and it's reasonable to postulate that the amount of energy would vary roughly in line with our volumes – that is, with the cube of our size. However, since we are warm-blooded creatures, we also have to keep our body temperature constant. The amount of heat we lose from our bodies is proportional to our surface area. Therefore, the energy required to keep warm would vary with the square of our size. Hence, we might speculate that our energy use would vary somewhere between the square and cube of our size. Instead of energy use, scientists talk about *metabolic rate,* the amount of energy we use per day. We might make the hypothesis that heat loss is proportional to $M^{2/3}$ (where M is mass), and thus metabolic rate should be proportional to M^c for some constant c lying between $\frac{2}{3}$ and 1. A scientist called Max Kleiber did detailed experiments with a huge range of (warm-blooded) animals looking at their food intake and comparing to their mass. He found what is now known as Kleiber's Law, that metabolic rate is proportional to $M^{3/4}$. If we scale our human by a factor of k in size, the mass will be scaled by k^3, so we can re-state Kleiber's law for different scale humans to say that the metabolic rate is proportional to $k^{9/4}$.

Suppose Gulliver eats 2,500 calories a day to maintain his weight. The Lilliputians will need to eat $\left(\frac{1}{12}\right)^{9/4}$ times as much, or just 9.3 calories a day. A 2019 paper on this topic [4] suggests that in fact their calorific needs would be more like 57 calories, based on Quetelet's observations about how mass changes with height, but that just amplifies the problem we are about to describe. That problem is the following: Gulliver tells us that everything in Lilliput is smaller than in England, with the same 1 inch to 1 foot scaling. The trees are one twelfth the height, the sheep and cows the same, as well as all the crops. But the calories contained in a given food are proportional to its mass. 10g of butter contains ten times as many calories as 1g of butter. A normal-sized apple contains 100 calories. So, 25 apples a day would provide Gulliver with his daily energy requirements. But Lilliputian apples are only $\frac{1}{1728}$th the mass of normal apples, meaning a Lilliputian would have to eat over 160 of them to meet their energy requirements! In general, the Lilliputians have to eat proportionately $\frac{(1/12)^{9/4}}{(1/12)^3} = (12)^{3/4} \approx 6.45$ times as much of their tiny food as Gulliver needs of his. They would have to spend all day eating!

Getting Wet

All liquids have *surface tension*. This is what gives liquids in a test tube that meniscus, what allows raindrops and bubbles to exist, and what makes bubbles pop if they get too big. Surface tension is an intrinsic property of a given liquid, like density, and is independent of scale. Immerse an object in water and when it comes out it will be covered in a thin film of water, about half a millimetre thick -- this thickness arises from the surface tension and adhesive properties of water, and so is independent of the size of the object. Taking 1.8m^2 as the average body surface area of an adult, and remembering a cubic centimetre of water weighs 1 gram, the estimated weight of the water you carry when you get out of the bath is $0.05 \times 18{,}000 = 900$g, a negligible amount. The Lilliputian's surface area is $\frac{1}{144}$ ours, so the weight of water they carry will be $\frac{1}{144} \times 900 = 6.25$g. Adult humans weigh about 75kg on average (women less, men more), so adult Lilliputians weigh $\frac{75}{1728}$kg $= 43.4$g. The water is about 14% of their weight – analogous to us suddenly having to carry an extra 10kg around. It's even worse for the kids in *Honey I Shrunk the Kids*, who are 1/200th our size. They weigh 0.0094g, and the water round them would weigh 0.023g – more than double their weight. They would likely drown.

Raindrops, which are nothing to us, could be quite concerning for miniature creatures. A typical 5mm diameter spherical raindrop, since the density of water is 1, has mass $\frac{4}{3}\pi(0.25)^3$ grams, which comes out to 0.065g, around 0.15% of a Lilliputians weight. Proportionately, it'd be like us getting repeatedly pelted with apples. But the poor 1/200 scale shrunken kids would be in mortal peril, as a single raindrop would be seven times their weight!

The Animal Kingdom

Fantastical animals like Pegasus, King Kong, and Hagrid's giant spider Aragog, are a common feature of fiction. Since the range of sizes of real animals is so vast, could creatures like this exist? As an indicative example, mammals range in size from the tiny bumble-bee bat (weight 1.7g, length 2.9cm) to the blue whale (the largest confirmed specimen of which was 29.9 metres (98 feet) and 190,000kg). Given the effects of the Square-Cube Law, how is this possible? The answer is that, of course, large mammals are not simply small ones scaled up (unfortunately for King Kong). All mammal bones have similar strength, but the bones of larger mammals are proportionately thicker. Compare the legs of an elephant to those of a mouse. Another difference between large and small mammals is their gait. When horses run, their legs are always completely straight when they touch the ground – this is the position of maximum strength. If they bent, then their legs would buckle at the knee during a gallop. Cats and other small mammals can be much more cavalier about such things, and their legs can remain bent during motion.

Kleiber's law tells us that the smaller the mass of an animal, the more it has to eat as a proportion of its mass. Small mammals lose heat much more quickly in proportion to how much heat they generate. To counter this, small mammals are usually covered in thick layers of fur, they have a more spherical shape (a tiny mouse is a fluffy ball compared to a much less fluffy, longer, leaner rat), and they do not exist in colder climates. There are Arctic hares and Arctic foxes, but no Arctic mice. The young of various mammals are also protected. For example, grey seal pups are born with a coat of long white fur which they lose at about 4 weeks of age when they are big enough to sustain their body temperature without it. At the other extreme, elephants have the reverse problem of getting overheated. This is one reason why their ears are so big: it increases their surface area and allows them to lose heat more quickly.

In Greek mythology there is a winged horse called Pegasus. Experiments with birds have shown that there is an upper bound on ability to fly, in terms of what's known as the wing loading, which is given by the equation wing loading $= \frac{\text{mass}}{\text{wing area}}$. It has been shown that the upper limit for wing loading is about 25. If we simply double all a bird's linear dimensions, the mass will increase by a factor of 8 and the wing area by a factor of 4, so the net effect is that the wing loading doubles. Quite soon we reach a point where the wing loading exceeds 25, and the bird cannot fly. In practice, larger birds have a different shape from smaller birds, with proportionately larger wingspans and smaller bodies. If we assume that Pegasus weighs about what a normal horse weighs, then the upper bound of 25 for wing loading would require him to have a wingspan of approximately 40 metres. He will need a big stable! There is also the issue of the power required to gain sufficient flight speed. Divine intervention aside, an angel would have to look pretty strange in order to be able to take off from the ground and fly. The biologist J.B.S. Haldane, in his famous 1927 essay 'On being the right size' [5], estimates that their chests would have to protrude four feet in order to contain the musculature required to move their (necessarily enormous) wings. Humans can glide, of course, with artificial wings, but that is different as no motor power is required.

Insects and Arachnids

Insects and arachnids (such as spiders) are cold-blooded, which means they do not have the heat-loss challenges of mammals. This allows them to be very small, but why don't we see very large insects? The heaviest recorded adult insect is the Tree Weta, which can grow to 20cm long and a weight of 71g. The goliath bird-eater spider is the heaviest spider around, at 175g, with a length of 13.1cm. Of course, if we simply scaled ants up to the size of humans, or even bigger, as in the 1954 movie *Them*, then they would collapse under their own weight in the manner of other giants. But nature doesn't simply scale up. Assuming thicker legs and other adaptations, what's stopping there being six-foot beetles, or other horrors? One reason is that insects and arachnids have their skeletons on the outsides of their bodies (exoskeletons). This means that as they grow, they must periodically shed these hard skins. While the new skin is hardening the creature is very vulnerable to attack, but also, above a certain size, would simply collapse under its own weight, with no hard skeleton to support it. Another factor is that, like all animals, insects and arachnids need oxygen to survive. They take in oxygen directly from their body surface, and it travels to their cells through little tunnels called tracheae. Air moves through these tracheae via tiny holes called spiracles. Consequently, the amount of oxygen insects can absorb is directly proportional to their surface area, but of course the number of cells is proportional to their mass, which is proportional to volume. So, the Square-Cube Law again comes into play; at a big enough size, not enough oxygen could be taken in, and the insect would suffocate. To grow larger, other kinds of animals have developed different ways of getting oxygen to their cells. In mammals, birds and so on, blood circulates, pumped by the heart, carrying oxygen in blood vessels around the body. The oxygen gets into our blood through our lungs, which have an extremely large surface area – the surface of the lung has many tiny folds precisely for that reason.

Could Aragog, the giant spider from the Harry Potter books, exist (without magic, that is)? Aragog is about the size of a car – two to three metres. If we assume that spider skeletons, like ours, can take up to ten times the normal pressure, then a scaled-up spider would collapse under its own weight at anything over a scaling factor of ten. Even a ten-times scaled up goliath bird-eater would be only 1.31m long. Aragog would also not be able to breathe. Research in 2005 at the University of California showed that that insects could cope with oxygen concentrations one fifth of what's in the atmosphere (and in fact can block their tracheae selectively to keep oxygen levels down, to reduce oxidative stress in their bodies). Thus, by keeping his tracheae permanently open, a giant spider five times normal size could still just about breathe. Based on this, the maximum possible spider size would be five times the goliath bird-eater, or 66cm long.

Prehistoric insects grew much bigger than they do now. The largest that we have evidence for are the Meganisoptera, giant creatures resembling dragonflies, such as the Meganeura Monyi from 300 million years ago. One fossil specimen had a wingspan of 71cm, and an estimated mass of over 200g. Part of the reason this large size is possible is because there was a higher percentage of oxygen in the air at several points in the earth's past, reaching 30% of the atmosphere at one point compared to 20% now. The concentration of oxygen matters because it means more can enter the same size trachea. However, even though oxygen increased in the Jurassic period, insect wingspans seem to have decreased. This could be because flying birds and reptiles evolved at this time. These could grow much bigger than even the biggest insects, which then risked becoming prey. It seems that to adapt to this, insects became smaller, so as to be a less enticing snack for any passing pterosaur. Another reason why we no longer have even moderately large insects is competition. Warm-blooded animals cannot grow very small, as we have already seen, but they can grow large. Thus, at the larger sizes, insects and arachnids would be in direct competition with other kinds of animal. But at smaller sizes, the field is wide open. It makes sense, therefore, that insects and arachnids would evolve into this niche. The Weta exists only in New Zealand, where the largest native mammal is a bat. Once mice were introduced, the Weta quickly became endangered.

Applications

The ramifications of the Square-Cube Law go beyond fictional worlds and the animal kingdom. They affect many engineering problems, from the challenges of building skyscrapers to aircraft design. We finish our analysis of large and small with a handful of examples.

Buildings

We can make architectural models out of cardboard, but not the buildings themselves. Why? Because the Square-Cube Law tells us that if we double the linear dimensions of a building, the cross-sectional area of its supporting structure will be quadrupled, while its weight will be multiplied by 8. That is, the pressure on the structure will double. A material such as concrete that is perfectly fine for a one-storey building, will not suffice for a 100-storey skyscraper! Steel, which is over twenty times stronger than concrete, is a better choice. Once we are using the strongest possible construction material, as we reach the limit of height for a given shape, the next adjustment that must be made is to increase the area at the base to give more support. It's no coincidence that the Pyramids are pyramid shaped.

Aircraft
Large aeroplanes cannot simply be scaled-up versions of smaller ones. At the time of writing this, the smallest and largest current commercial passenger planes are, respectively, the Boeing 737 and the Airbus A380. The Airbus is just over twice the length of the Boeing and about 9 times the weight. If we just doubled all the Boeing's dimensions, its wing area would only increase by a factor of four, which would not be sufficient to carry the eightfold weight increase. In practice, the Airbus wings have a much bigger surface area, they are fatter, and that is how they can carry that extra weight.

Models and Prototypes
A working scale model does not guarantee a working full-size version, and vice versa. A famous example of this was Glasgow University's model of the Newcomen Steam Engine. This engine, designed by Thomas Newcomen, was widely used, mainly for pumping water out of mines. A cylinder was repeatedly heated with steam and then cooled with cold water, condensing the steam and hence creating a partial vacuum, that caused the piston in the cylinder to move. It worked, but not very efficiently, because a lot of heat was lost. The Glasgow scale model didn't work, and they gave it to a young instrument maker by the name of James Watt to fix. Watt realised that because of the Square-Cube Law, in the smaller version the problem was exaggerated, making the model not work at all. (The reason is that heat produced is proportional to volume, but heat lost is proportional to surface area. Decreasing the size decreases the volume more than the surface area.) In trying to fix this he came up with the idea of a separate condenser which was a huge advance in steam engine design and a milestone in the industrial revolution. The converse of this issue is that if you are designing a cooling system for a power plant, or similar, and it works in your prototype scale model, that does not guarantee it will work in the full-size version, as the surface area will be relatively smaller as a proportion of volume.

Flatland and Beyond
Our next starting point is a curious book published in 1884, entitled *Flatland: A Romance of Many Dimensions* [6]. Like Gulliver's Travels, it is a satire, but this time, the narrator is not from our world. He is A. Square, an inhabitant of Flatland – an entirely two-dimensional realm. "A. Square" is in fact Edwin A. Abbott. He was a teacher and clergyman – headmaster of the City of London School, a boys' school in central London, at which he had also been a pupil. (About 140 years later I went to the corresponding girls' school.) *Flatland* was popular at the time but has also been very influential since, leading to many sequels and spin-offs.

The descriptions of Flatland's rigid class structure, dogmatic priestly class, and the role given to women in that realm, are a satire on the Victorian society in which Abbott lived. He was strongly in favour of improving education for women, and an advocate for a more tolerant 'Broad' Church, that rejected ritualism and favoured individual choice in terms of Christian worship. As well as the social satire, the book has another aim. The struggles of A. Square to understand the possibility of a third dimension, from his two-dimensional worldview, are intended to aid us in our own appreciation of the possibilities of the fourth dimension, trapped even as we are in a three-dimensional world.

Flatland is a short book of around a hundred pages, split into two parts. In Part I, we are given a tour of Flatland, its conventions and society, which take to extremes some of the worst aspects of Victorian society. A. Square tells us that in Flatland, the men are all polygons, and the women are lines. The lowest social level are the narrowest isosceles triangles – these being the closest to women, I suppose. "Our Soldiers and Lowest Classes of Workmen are Triangles with two equal sides, each about eleven inches long, and a base or third side so short (often not exceeding half an inch) that they form at their vertices a very sharp and formidable angle." The middle classes are equilateral triangles, the “Professional Men and Gentlemen” are Squares and Pentagons, with the nobility consisting of Hexagons and higher sided polygons (all regular). Social mobility is possible. “After a long series of military successes, or diligent and skilful labours, it is generally found that the more intelligent among the Artisan and Soldier classes manifest a slight increase of their third side or base, and a shrinkage of the other two sides.” The bigger the angle, the bigger the brain. If you work hard and don’t disgrace yourselves, your sons may have bigger angles (approximately half a degree each generation), until the wonderful day when an equilateral son is born. From that point on, each successive generation produces regular polygons with one more side than their parents. So, an equilateral triangle’s son is a square, his grandsons are pentagons, and so on. What stops triangles gradually ceasing to exist is that the fertility rate gradually declines as the number of sides increases. Regularity can be disrupted by bad behaviour – your crimes are visited upon your descendants. (Children who are born irregular can sometimes have this corrected at the Circular Neo-Therapeutic Gymnasium.) The higher classes may have many hundreds of sides, and once you get beyond a certain point, you are referred to out of respect as a circle. “It is always assumed, by courtesy, that the Chief Circle has ten thousand sides.” The irregular classes are stupid and venal because of their irregularity – they can’t help it. “Why blame the lying, thievish Isosceles when you ought rather to deplore the incurable inequality of his sides?” But this shouldn’t stop them being punished, naturally. “In dealing with an Isosceles, if a rascal pleads that he cannot help stealing because of his unevenness, you reply that for that very reason, because he cannot help being a nuisance to his neighbours, you, the Magistrate, cannot help sentencing him to be consumed [executed] – and there’s an end of the matter.”

Houses in Flatland are regular pentagons, the angles of triangles and squares being sufficiently sharp that buildings of these shapes pose a health hazard to the unwary traveller. Women, because they can make themselves almost invisible by approaching point-on, must at all times when out of doors utter a peace-cry so that their approach can be heard. All houses, by decree, must have separate entrances for men and women, to avoid any accidental impalements when arriving or departing.

Given the rigid hierarchy, it is imperative that one be able to tell the different polygons apart. But how is this possible, when all one can see is lines? The answer is that the atmosphere is slightly foggy, so that one can perceive depth. Therefore, a hexagon would appear as a central line with a receding line on either side, and could be distinguished from, say, a pentagon, by the fact that the lines on a hexagon recede less than the pentagon’s lines would. Only triangles and squares can appear to be lines (women!), and even then, only from certain viewpoints. Since one’s social standing is completely determined by the number of one’s sides, anyone born irregular is a great threat to society. If you saw an angle of 120 degrees approaching, and took him to be a regular hexagon, and even invited him into your home, imagine the horror if it turned out you had been speaking to an irregular quadrilateral! Naturally, explains Square, such aberrations must be destroyed at birth. Part I ends with a discussion of how women are treated in Flatland, in particular bemoaning their lack of education. In the concluding sentence, Square makes a “humble appeal to the highest Authorities to reconsider the regulations of Female education” (a cause very much espoused by Abbott himself in the real world).

In Part II, the narrator is visited by a stranger (“Sphere”) from Spaceland, a three-dimensional world, and discovers to his shock that there are more than two dimensions. He visits ‘Lineland’ and ‘Pointland’ in dreams, and Sphere takes him to Spaceland.

Square first dreams of Lineland. The whole world is a line; men are line segments and women are points. It being impossible to pass each other on a line, the inhabitants of the world spend their lives with the same neighbours. They see only points and distinguish their fellow creatures by sound. Square attempts to explain to the Monarch of Lineland that there is another dimension, by moving in and out of Lineland, but of course all the Monarch can see is a point that appears and disappears.

Later in the book, in another dream, Square visits Pointland, where the King, who is the whole of the universe he rules, cannot conceive even of the existence other beings, so interprets Square's voice as being another aspect of his own thoughts.

One evening, a mysterious stranger suddenly appears in Square's living room – this is Sphere, a visitor from Spaceland. Square, seeing a circle, assumes that a member of the priestly class has entered his home. But then Sphere moves through Flatland, in the same manner that Square moved through Lineland. Square sees a circle becoming smaller and smaller, diminishing to a point and finally vanishing. Then Sphere explains how he entered Square's house. Not through the doors – they are closed – and not through the roof, but from "above". We three-dimensional creatures can see and move through the insides of things in Flatland, just as an inhabitant of a four-dimensional universe could see inside us and move in and out of closed spaces without difficulty. Square still not being convinced by this "magic trick", Sphere endeavours to persuade him by means of analogy.
"We begin with a single Point, which of course – being itself a Point – has only *one* terminal Point. One Point produces a Line with *two* terminal Points. One Line produces a Square with *four* terminal points. […] 1, 2, 4 are evidently in Geometrical Progression. What is the next number?"

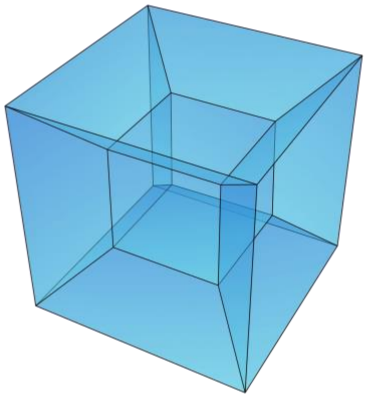
Square correctly replying "Eight", Sphere replies "Exactly. The one Square produces a *something-which-you-do-not-as-yet-know-the-name-for-but-which-we-call-a-Cube* with *eight* terminal Points. Now are you convinced?"

Sphere also points out that if we define the "sides" of a shape as the component parts of dimensionality one less than the shape, that bound the shape, then a Point has 0 sides, a Line has 2 "sides" (the two extremal points), a Square has 4 sides, so following the pattern 0, 2, 4, … the "Cube" would have 6 "sides", and indeed it does have six square faces.

Eventually, Sphere takes Square outside of Flatland to look on it from above, and Square becomes a convert to the Gospel of Three Dimensions. But when he asks Sphere to show him the Land of Four Dimensions, Sphere utterly rejects such an idea as being entirely inconceivable. Square persists – surely in the fourth dimension a moving cube can produce a four-dimensional solid with 16 terminal points (vertices) and 8 "sides" (each of which would be a cube)? And what about a fifth or higher dimensions? This does not convince Sphere, who angrily returns Square to Flatland, where he is imprisoned for preaching his heretical theories. How arrogant and ignorant the King of Lineland is, for believing there could be only one dimension. How arrogant the Flatlanders have been for believing in only two. The inescapable inference is that any Spacelander who rejects the idea of a fourth dimension is being similarly blind to the obvious.

Why did Abbott write Flatland? It is completely different from his other books, around fifty of them, which were mostly school textbooks, or books on theology or literature. Abbott was not a mathematician, but a classicist and theologian. By all accounts he had a good education in mathematics. One of his mathematics teachers had been Robert Pitt Edkins, Gresham Professor of Geometry from 1848-1854, and all Classics students at Cambridge had to pass the mathematical Tripos. Through his life, there were scientists and mathematicians in his social circle, including Charles Hinton (though we don't have firm proof that they met), whom we'll encounter presently. We know, for example [flatterland, see below] that Abbott and the physicist John Tyndall were dinner guests at George Eliot's house on the same night, and could easily have discussed the fourth dimension, which was a very hot topic at the time. Nowadays, we are quite happy to work with multidimensional spaces, each additional dimension being thought of as an additional quantity associated to an object. For example, when modelling the weather, we might, for each point in the atmosphere, assign seven numbers: the latitude, longitude, height above sea level, wind speed and direction, temperature, and air pressure. We can therefore think of this as a seven-dimensional "space". But it took a very long time for mathematics to become comfortable with such ideas. The very word geometry comes from measuring the earth, and for the early Greeks, geometry consisted of the geometry of the plane. If you are drawing up plans or dividing land in a will, you certainly can approximate any bit of the earth as a plane, so this served well for practical purposes. Later, geometry of three dimensions (or "solid geometry") came along, and for astronomical purposes the geometry of spheres was needed. But by the nineteenth century several different currents of thought led to the realisation that other kinds of geometry were worth considering. Attempts to prove the parallel postulate from the other four axioms of Euclid eventually (after well over a millennium) led to the uncomfortable realisation that actually it was logically independent from them. During the 19th century people like Gauss (1777 – 1855), Bolyai (1802 – 1860) and Lobachevsky (1793 – 1856) gave explicit proofs that there were geometries, called Non-Euclidean geometries, where this occurs – one example is the surface of a sphere, which is a two-dimensional surface where "lines" are actually great-circles, the parallel postulate fails, and as a result the angles in spherical triangles add up to more than 180 degrees. Meanwhile in physics, things like electromagnetic fields were being studied, which naturally give rise to settings where every point has a position but also a field vector – in essence a six-dimensional space. Another very useful tool in mathematics and theoretical physics was the 'quaternion' algebra invented by mathematician William Rowan Hamilton (1805 – 1865). Every quaternion is a combination of four independent quantities, $1, i, j$ and k, with rules to tell you how to multiply them (for example $i^2 = j^2 = k^2 = -1$, $ij = k, jk = i, ki = j$). But this is an algebra of four dimensions! In explaining these ideas of higher dimensions, a number of scientists, including German physicist Hermann von Helmholtz (1821 – 1894) had invoked the analogy of a two-dimensional being living on some mathematical surface like a plane, sphere or torus, trying to understand its universe. An important precursor to Flatland is an article "What is the fourth dimension?" by Charles Howard Hinton, a mathematician, teacher and writer (1853-1907), which was almost certainly read by Abbott. It asks the reader to imagine "a being confined to a plane", and to suppose "some figure, such as a circle or rectangle, to be endowed with the power of perception", just as A. Square is in Flatland. When the article was reprinted as a pamphlet in 1884, the publisher added the subtitle "Ghosts Explained", because of the suggestion that one explanation for ghosts appearing and disappearing, or passing through walls, could be that they were simply moving around in the fourth dimension. There's actually a remark to this effect in Oscar Wilde's 1887 haunted house parody *The Canterville Ghost*. "There was evidently no time to be lost, so, hastily adopting the Fourth Dimension of Space as a means of escape, he vanished through the wainscoting, and the house became quite quiet."

Within 25 years of *Flatland*'s publication came Minkowsi's formulation of four-dimensional space-time, where time is added to the three spatial dimensions. This is an excellent framework within which to understand Einstein's theory of relativity. Nowadays, in fact, string theorists tell us that our universe may really be ten-dimensional, or even 23-dimensional. One example of a four-dimensional figure is the hypercube, which we make by analogy. Just as a square is made by joining the vertices of two parallel lines, and a cube is made by joining the vertices of two parallel squares, a hypercube is drawn by joining the vertices of two parallel cubes (in the fourth dimension). The diagram, being a two-dimensional projection of a four-dimensional object, necessarily distorts some lengths, but we must imagine all the edges to be equal in length, all the 2-dimensional faces to be squares, and all the 3-dimensional faces to be cubes. We can see that this figure has 16 vertices (8 in each of the two initial cubes), 32 edges (12 in each of the two initial cubes plus another eight joining the corresponding vertices of the two cubes), 24 square 2-dimensional faces (the six each of the two initial cubes plus twelve made by joining corresponding edges of the initial cubes) and 8 cubic "solid faces".



A hypercube (image credit: Wikipedia)

A 4-dimensional sphere would have the property that each of its cross-sections is a sphere. We can give the equation of such a hypersphere quite easily: $x^2 + y^2 + z^2 + t^2 = 1$. But we don't have much hope of visualising it! Both these ideas can be extended to an arbitrary number of dimensions. One can also define 4-dimensional, or n-dimensional, pyramids. For example, in four dimensions we could have a cube-based pyramid where every vertex of a cube is joined to a fixed new vertex, creating a shape with one cubic "face" and six "faces" that are square-based pyramids.

The ideas of Flatland have been explored and developed by many authors. In 1907, Charles Hinton published "An Episode of Flatland: How a Plane Folk Discovered the Third Dimension" [7]. His is a different "Flatland" – the inhabitants live on the surface of a circular planet in a 2-dimensional universe. More recently, there are Dionys Burger's 1957 novel *Bolland* [8] (Sphereland in English), A. K. Dewdney's *The Planiverse* (1984) [9], Rudy Rucker's *The Fourth Dimension* (1985) [10] and *Spaceland* (2002) [11], and Ian Stewart's *Flatterland* (2001) [12]. Ian Stewart (a former Gresham Professor of Geometry) has also produced an excellent *Annotated Flatland* [13] which is highly recommended, and which was the source of much of the biographical material on Abbott and Hinton given here. See the list of recommended reading at the end of this transcript for details of these books.

I'll restrict my remarks here to a brief discussion of *The Planiverse*, because this is probably the most serious attempt to think carefully about how life on the plane might actually work. In *The Planiverse*, a group of students and their professor (Dewdney) make a computer simulation, "2DWORLD" of a two-dimensional universe. They create a planet, Astria (a reference to Hinton's 'flatland' planet), peopled with simple beings called FEC's (Ffennel-Edwards Creatures, after the students who designed them), who spend their time wandering about catching and eating other creatures called *throgs*. But one day, due to some kind of universal morphic resonance, they start to be able to see another 2D world, that they didn't create, and communicate with a being called Yendred who lives there. Over the course of several months, he shows them his planet, Arde, and they learn about the life of the Ardeans, before Yendred one day breaks contact for unexplained reasons, and is never seen again. In the introduction to the 2001 reissue of the book, Dewdney claims that some people believed the account was real, in spite of the many tongue-in-cheek references and in-jokes it contained. I'm not sure, though, if this claim itself was a joke. It's obviously no coincidence that "Yendred" is (almost) Dewdney spelt backwards. One of the research students is named Alice Little – surely too much like Alice Liddell to be an accident. Arde has a sister planet Nagas (Sagan backwards), and there are numerous other such incidences.

The Planiverse is a two-dimensional universe: its natural laws are different from ours. For example, it makes sense for gravity to have an inverse linear law, rather than an inverse square law. This preserves, for example, what we observed earlier that for a fixed density, the gravitational acceleration experienced on the surface of a planet is proportional to its radius, because the mass of a circular planet is proportional to the square, rather than the cube, of this radius. Chemistry would also be different. A molecule that cannot exist in the plane, cannot exist. Also, there is a more extreme version of the chirality phenomenon that we have – many molecules will have a different mirror image. In terms of biology, nothing like a human body could exist. Something like a digestive tract passing all the way through a body would cause it to fall into two parts. Ardean bodies have to have exoskeletons, like insects, not endoskeletons, because an internal skeleton would prevent the flow of bodily fluids – oxygen, nutrients etc. Since any tube would break the body into pieces, fluids pass through the body by means of “zipper organs”, that open and close to allow a bubble of fluid to pass through the body.

The book does an impressive job of considering how life would be for the Ardeans. Arde is a circular planet, and the creatures living on its surface know only two directions: east-west and up-down. An Ardean cannot turn his head in the same way we can. From looking east, they can move their heads upwards and swing round in a semicircle to point west. Their entire visual world is a single line. When moving around the surface of the planet, their two directions are east-west and up-down, so if two people travelling in opposite directions meet, one must climb over the other. In cities they have passing places – pits that close at both ends where a group of (say) eastward travellers wait to let some westward travellers pass; then the doors are opened, and the eastbound travellers can proceed. To allow free travel on the surface, houses and other buildings are underground, with “swing stairs” - staircases that lift up and down to let people pass doorways. Stairs in the houses also must be moveable, because otherwise they would break the house into separate unreachable compartments. Similarly, doors cannot be built into supporting walls, or every time the door opened the house would collapse. There is a constant risk of creating a partial vacuum or getting trapped in a sealed area where the air will quickly run out.

Art, music and sport are discussed. They play a game like volleyball, but of course a net is not possible, meaning the central pole dividing the teams is opaque, and the ball suddenly appearing over it is more of a surprise. In terms of technology, they have electricity but obviously wiring a house would split it into separate parts that you couldn’t cross between, so they use batteries. On building: “Nails are useless, since they part any piece of material they are driven through. Saws are impossible. A beam could only be cut with something like a hammer and chisel”. As a result, buildings are mainly constructed using glue. Many other details on machines like steam engines and internal combustion engines are described in the appendix to the book. The ingenuity displayed is impressive.

Flatland continues to inspire – *Flatland: The Movie*, starring Kristin Bell, Martin Sheen and Michael York, received positive reviews when it came out in 2007. Thankfully, the story is modified so that women are also polygons, rather than lines. It was followed by a sequel, *Flatland 2: Sphereland*, in 2012, based on Dionys Burger’s *Bolland*.

Mathematical Metaphors in Fiction

We’ll finish with a whistlestop tour of mathematical metaphors in fiction. If a writer is comfortable with mathematics (which, unfortunately, seems unusual), then it will naturally enter the armoury of knowledge they have at their disposal for constructing metaphors. We discuss, very briefly, three such authors. An obvious omission from this list is C. L. Dodgson, the mathematician better known as Lewis Carroll. This is because it felt superfluous to talk about him given the excellent 2019 Gresham lecture by our former Geometry professor, and Carroll expert, Robin Wilson, which is available online [14]. My feeling with all these authors is that they don’t deliberately set out to be mathematical; it’s more that, if you come down with a case of mathematics, you can’t help yourself – it enriches everything you do.

George Eliot

Mathematical allusions abound in Eliot’s work. To give just one example, consider the description of Mr Casson, landlord of the Donnithorne Arms, given in Chapter 2 of her first novel, *Adam Bede* [15].

'Mr Casson's person was by no means of that common type which can be allowed to pass without description. On a front view it appeared to consist principally of two spheres, bearing the same relation to each other as the earth and the moon: that is to say, the lower sphere might be said, at a rough guess, to be thirteen times larger than the upper [...] But here the resemblance ceased, for Mr Casson's head was not at all a melancholy-looking satellite, nor was it a 'spotty globe', as Milton has irreverently called the moon.'

This striking metaphor is available to George Eliot because of her mathematical literacy – and in fact this number thirteen is a non-trivial calculation. To what does it refer? The diameter of the Moon is 3,474km, compared to the Earth's 12,742km, which means that their diameters are in the ratio 1 to 3.668 and their volumes are in the ratio 1 to 49.34. However, when we look at a picture of two spheres "on a front view" what we actually see is two circles. Thus, what the brain intuitively grasps is probably the ratio of their cross-sectional *areas*, not their volumes. This ratio is 1 to 13.45. (A worse approximation to the diameters, like 2,100 miles for the Moon and 7,900 miles for the Earth, would give a ratio of 1 to 14, so the fact that Eliot chose 1 to 13 is evidence that she knew these numbers quite accurately.)

George Eliot was the pseudonym of Mary Ann Evans (1819-1880). She was interested in mathematics, and studied it throughout her life, finding it a solace in times of stress. In an 1849 letter to friends, during a difficult time, she wrote, 'I take walks, play on the piano, read Voltaire, talk to my friends, and just take a dose of mathematics every day'. Adam Bede also seems to find reassurance in the eternal truth of mathematics, consoling himself after his father's death with the thought that 'the square o' four is sixteen, and you must lengthen your lever in proportion to your weight, is as true when a man's miserable as when he's happy'. For a much deeper look into Eliot's use of mathematics, Derek Ball's book is highly recommended [16].

Tom Stoppard

The playwright Tom Stoppard often makes use of mathematical ideas. In *Rosencrantz and Guildenstern are dead* [17], he plays with ideas of chance, probability and randomness. When they are betting on coin tosses in the first act of the play, the coin comes up heads 92 times in a row. Although any coin toss has an independent likelihood of coming up heads or tails, so this is just as likely as any other specified sequence of outcomes, we, with the protagonists, would find this particular outcome strangely unsettling and they conclude that they must be in the midst of some unnatural or supernatural events (as indeed they are, being minor characters in another play). Stoppard's wonderful 1993 play *Arcadia* [18] is bursting with mathematical ideas. Part of the story involves a mathematical prodigy, Thomasina Coverly, whom we meet in 1809; she is interested in how we can capture natural phenomena with equations – if there is a curve like a bell, she asks (a reference to the normal distribution in statistics), why not a curve like a bluebell?

"*I, Thomasina Coverly, have found a truly wonderful method whereby all the forms of nature must give up their numerical secrets and draw themselves through numbers alone. This margin being too mean for my purpose, the reader must look elsewhere for the New Geometry of Irregular Forms by Thomasina Coverly*."

The New Geometry conceived by Thomasina is the idea of shapes produced by repeated iterations – what we would now call fractals. Iterative processes can be used to produce surprisingly realistic images of plants, as well as more familiar examples of fractals, such as the Sierpinski triangle and the Mandelbrot set. There is also a nice mathematical in-joke to enjoy in this quote. The meanness of the margin is a reference to the famous marginal note made by Fermat in his copy of Diophantus's Arithmetica, that there are no positive integer solutions to the equation $a^n + b^n = c^n$ when $n > 2$. "I have found a truly marvellous proof of this", he wrote, "but the margin is too small to contain it".

<u>Herman Melville</u>
Mathematical references can be found in many, perhaps all, of Herman Melville's novels, but we focus here on *Moby-Dick* [19]. Any mathematician reading the novel will be struck by the number of mathematical references, and evident mathematical knowledge, shown by that book. Mathematical ideas creep in throughout, for instance in Ahab's praise for a loyal cabin boy: "True art thou, lad, as the circumference to its centre". We give just two further examples here and refer the interested reader to a forthcoming paper on this subject [20].

Here is Ishmael watching a whale swimming far off:

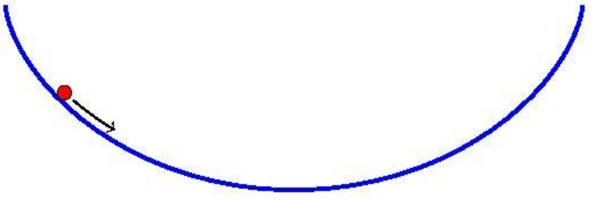
> '*Even if not the slightest other part of the creature be visible, this isolated fin will, at times, be seen plainly projecting from the surface. When the sea is moderately calm, and slightly marked with spherical ripples, and this gnomon-like fin stands up and casts shadows upon the wrinkled surface, it may well be supposed that the watery circle surrounding it somewhat resembles a dial, with its style and wavy hour-lines graved on it. On that Ahaz-dial the shadow often goes back.*'

This lovely 'fin as gnomon' metaphor is enhanced when we discover that the 'Ahaz-dial' refers to what is believed to be the earliest written reference to sundials, in the book of Isaiah. '*Behold, I will bring again the shadow of the degrees, which is gone down in the sun dial of Ahaz, ten degrees backward. So the sun returned ten degrees, by which degrees it was gone down'.* (Isaiah 38:8, King James version, see also II Kings 20:11) Ahaz was king of Judah in around 730-700 BC. The story tells how the shadow on the dial miraculously moved backwards, as a sign from God that he would cure the sickness of Ahaz's son Hezekiah.

The following quotation is an indication of how much deeper Melville's mathematical understanding went than standard school-level numeracy.

> '*Removing this hatch we expose the great trypots, two in number, and each of several barrels' capacity. When not in use, they are kept remarkably clean. Sometimes they are polished with soapstone and sand, till they shine within like silver punchbowls. During the night-watches some cynical old sailors will crawl into them and coil themselves away there for a nap. While employed in polishing them—one man in each pot, side by side—many confidential communications are carried on, over the iron lips. It is a place also for profound mathematical meditation. It was in the left hand try-pot of the Pequod, with the soapstone diligently circling round me, that I was first indirectly struck by the remarkable fact, that in geometry all bodies gliding along the cycloid, my soapstone for example, will descend from any point in precisely the same time.*'

Try-pot from wreck of whaling ship "Two Brothers", which sank in 1823 (public domain image by US National Oceanic and Atmospheric Administration)



Inverted cycloid – an object released from any point will take the same time to slide to the bottom (image by Sarah Hart)

The reference here is to the so-called tautochrone problem, to find the curve for which the time taken by a frictionless object sliding under gravity to the lowest point of the curve is independent of the starting point. A cycloid is the curve produced by a point on the circumference of a circle, or wheel, as it rolls along a straight line. If the circle has radius r, the line is the x-axis, and the point on the rim starts at the origin, then the cycloid consists of the points $r(\theta - \sin\theta, 1 - \cos\theta)$, where θ is the angle through which the circle has rotated. The first arch of the cycloid corresponds to the range $0 \leq \theta \leq 2\pi$. This shape, inverted, is the solution to the tautochrone problem to which Melville is referring. The tautochrone problem was first solved by Christiaan Huygens in 1659 – the required curve is a cycloid. The proof was given in his 1673 book *Horologium Oscillatorium.* If we take the inverted arch of the cycloid generated by a circle of radius *r*, then Huygens proved that the time of descent is $\pi\sqrt{\frac{r}{g}}$ (where *g* is acceleration due to gravity).

Conclusion

We have seen that fiction can be a great source of mathematical ideas that can lead us down fascinating paths. Mathematical metaphors can enrich good writing, and great fun can be had exploring the mathematical implications of imaginary worlds. In the next lecture, we look at how mathematical structures can be used in fiction, with examples drawn from authors including Georges Perec, Jorge Luis Borges and Eleanor Catton.

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