Equations that have changed the world

Chris Budd

GRESHAM COLLEGE

IMI

UNIVERSITY OF BATH

Equations are important

$$e^{i\pi} = -1$$

- They are eternally true
- They compress a huge amount of information into a single formula
- The same equation can have many different applications
- Equations lead to algorithms, which lead to new technology

Some famous maths equations

$$a^2 + b^2 = c^2$$ Pythagoras

$$a^n + b^n = c^n$$ Fermat

$$V + F = E - 2$$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Euler

Some famous physics equations

$$\mathbf{F}_{12} = \frac{G\, m_1\, m_2\, (\mathbf{x}_1 - \mathbf{x}_2)}{\|\mathbf{x}_1 - \mathbf{x}_2\|^3}.$$

Newton

$$i\hbar \frac{\partial \psi}{\partial t} = \left[\frac{-\hbar^2}{2m} \nabla^2 + V(\mathbf{r}, t) \right] \psi$$

Schrödinger

$$E = m\, c^2$$

$$G_{ab} = 4\pi\, T_{ab}$$

Einstein

Maxwell

My Five Favourite Equations

$$A\,x = b$$

$$A\,x = \lambda\,x$$

$$\begin{aligned} \hat{f}(\omega) &= \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt, \\ f(t) &= \tfrac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \hat{f}(\omega)\, d\omega \end{aligned}$$

$$\frac{Du}{Dt} = -\nabla P + \frac{1}{Re}\nabla^2 u, \quad \nabla . u = 0.$$

$$\frac{dx}{dt} = -\lambda\, x(t-\tau).$$

Equation one: The Linear Equation

$$A\,x = b$$

Most problems end up being this. Shopping, medical imaging, weather forecasting, ..

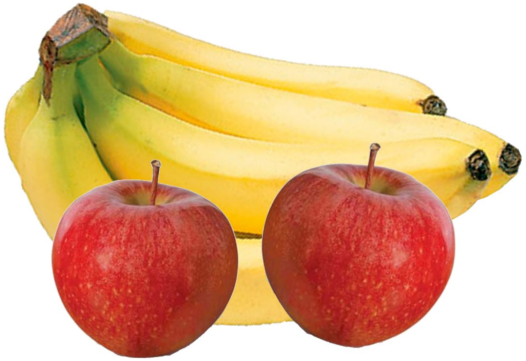
- **b** is known
- **x** is unknown
- A is a linear operator linking **x** to **b**

Examples of operators A:

Matrix: $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Differential: $$\frac{d^2x}{dt^2} \quad \text{or even} \quad \frac{\partial^2 x}{\partial y^2} + \frac{\partial^2 x}{\partial z^2}$$

Example: Going shopping



Want to buy x apples and y bananas

Apples cost £2 each. Bananas cost £3 each

Budget of £15

Each apple has 4 units of vitamin C, banana has 3 units

Total vitamin C content is 25 units

What values of x and y should we take?

Linear matrix equation

$$\begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 17 \\ 25 \end{pmatrix}.$$

$$A \qquad \mathbf{x} \qquad \mathbf{b}$$

Solution

$$\mathbf{x} = A^{-1}\mathbf{b}$$

$$A^{-1} = \begin{pmatrix} -1/2 & 1/2 \\ 2/3 & -1/3 \end{pmatrix}$$

$$\mathbf{x} = (x, y) \quad \text{where} \quad x = 4 \quad \text{and} \quad y = 3$$

Problem has two unknowns and is easy to solve

Problem with many N unknowns is much harder to solve

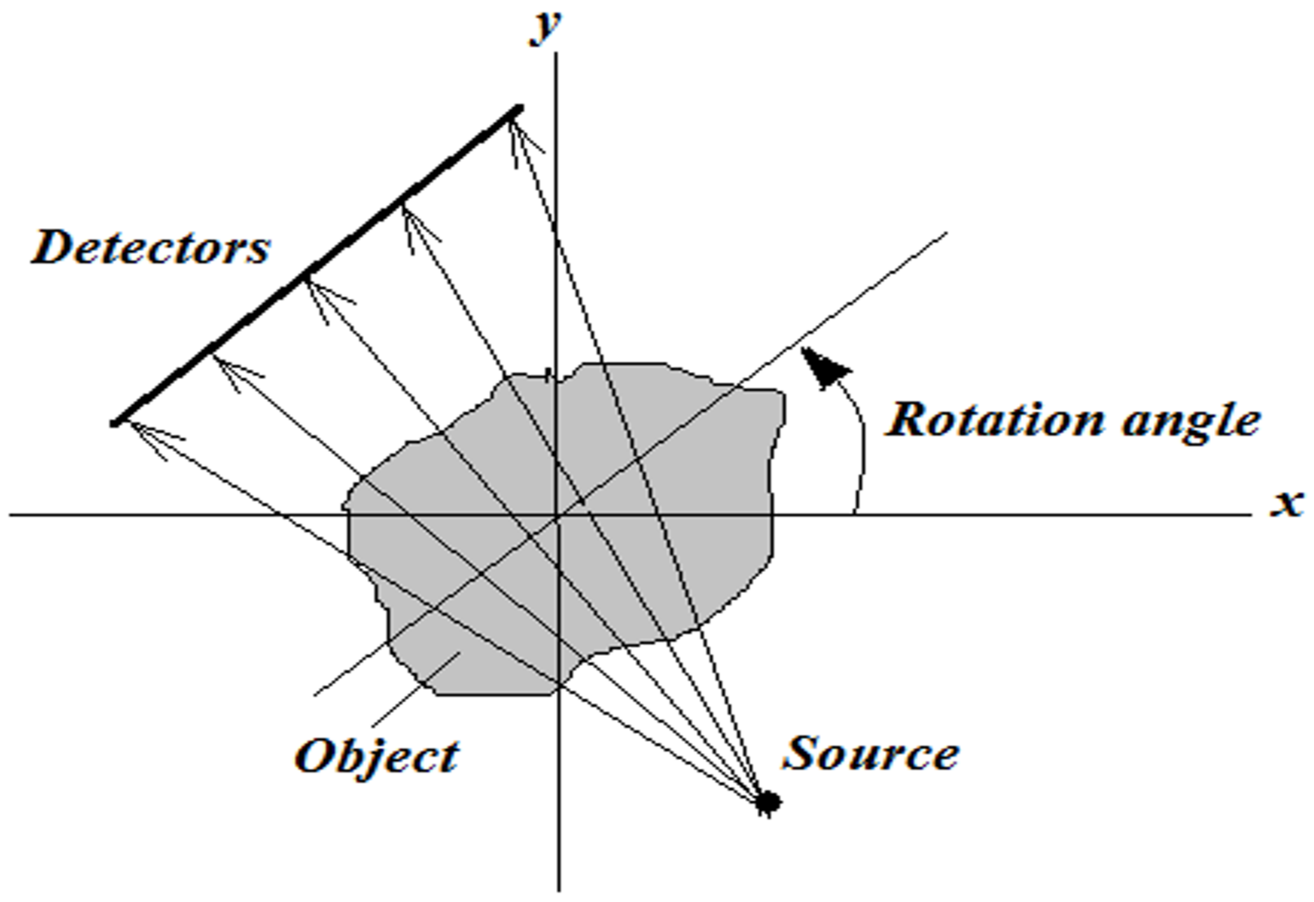
Matters as $A x = b$ has vast numbers of applications

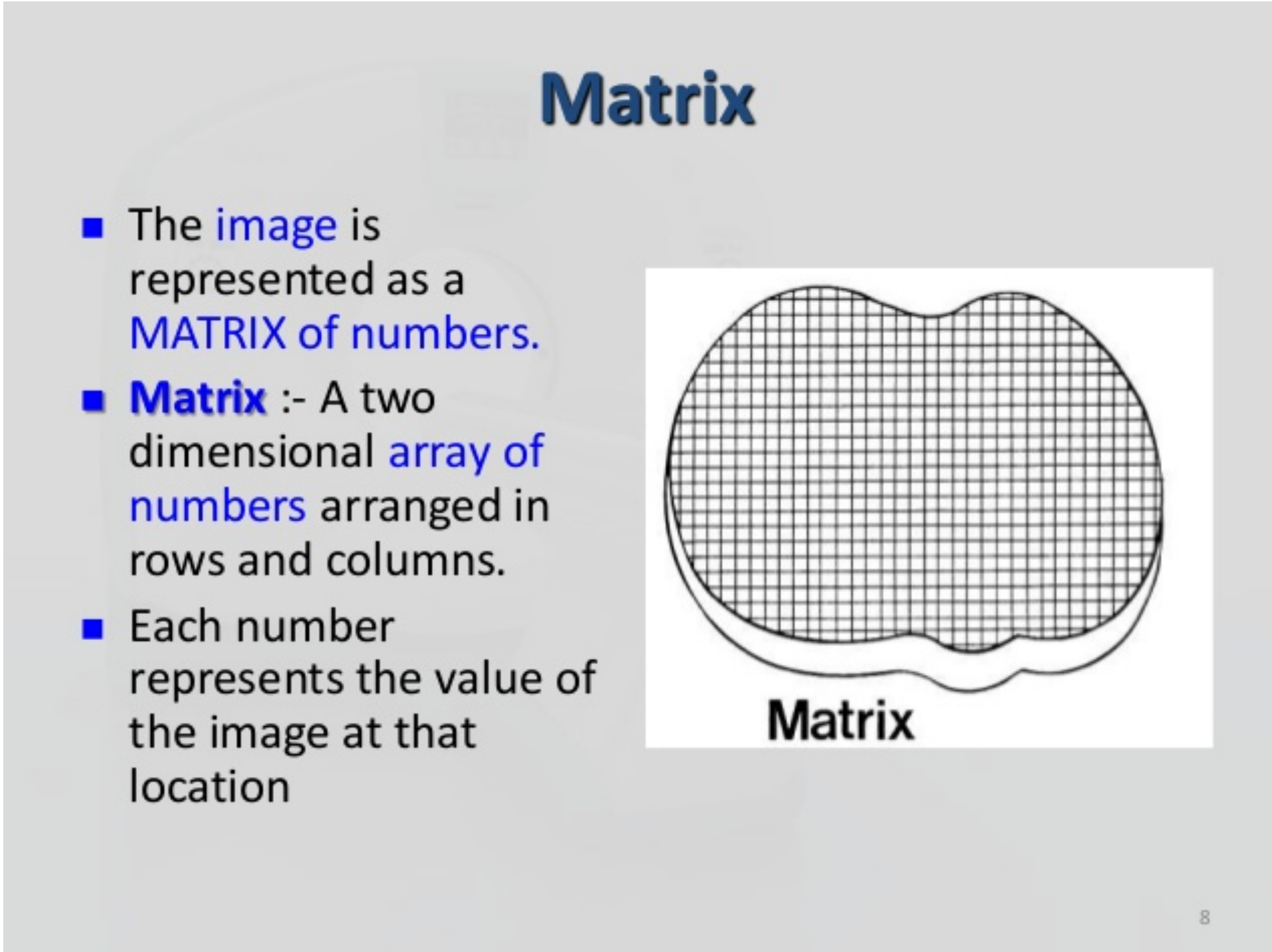
Eg. medical imaging , weather forecasting, banking, designing aircraft , building bridges, retail, …

N = 1000 000 0000 is not uncommon

Eg. Medical imaging

Shine many X-Rays through a body





x Organ optical density at each point (more than..)

b Attenuation of each X-ray

A Tomography matrix

Ill posed as more unknowns than equations

Use a-priori information (Bayesian)

How to solve it?

- Direct method:

Gaussian Elimination

- Iterative method: Successively improve the solution

Conjugate Gradient Method

- Fastest and best (but most complicated)

Multi-grid method

Equation 2: The Matrix Eigenvalue Equation

$$A\,\mathbf{x} = \lambda\,\mathbf{x}$$

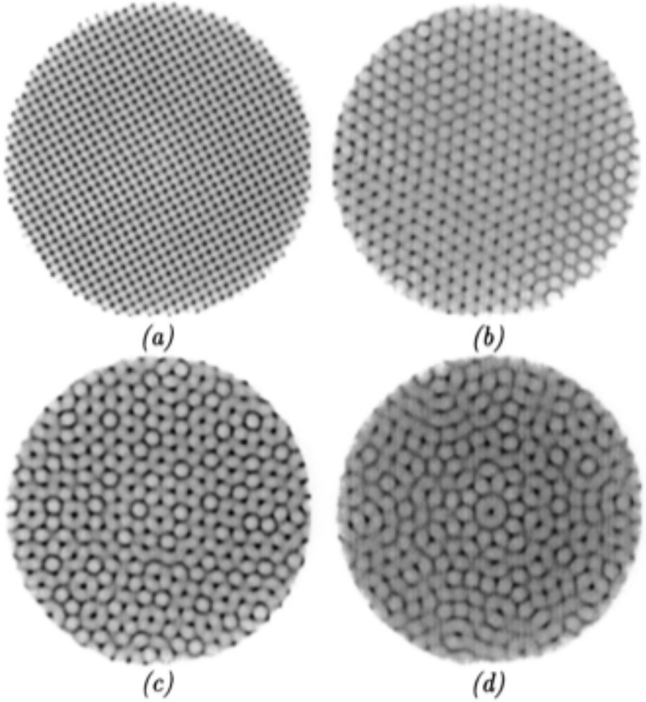
The equation for sound, WiFI and Google

$$\nabla^2 \mathbf{x} = -\omega^2 \mathbf{x}$$

Helmholtz Equation

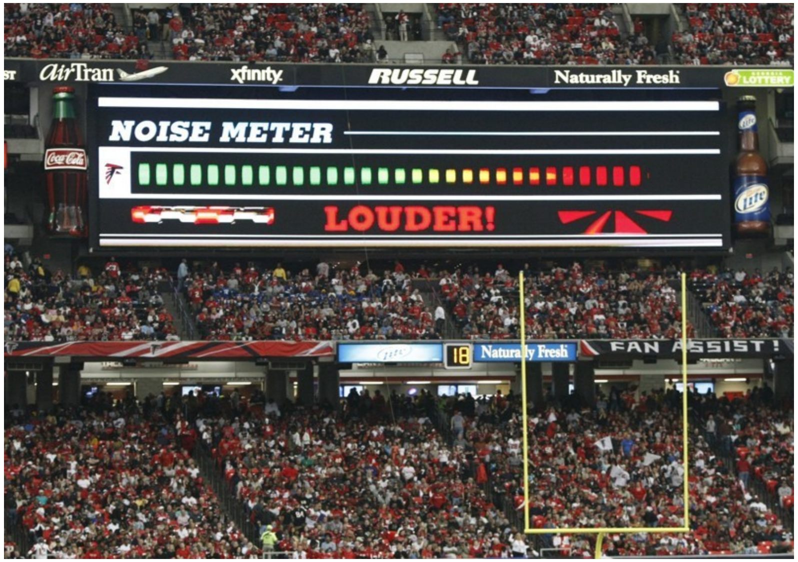
ω vibrational frequency

$\mathbf{x}$ vibrational mode



Vibrational waves in a dish

Singing in a football stadium



Example:

$$A = \begin{pmatrix} 2.5 & 0.5 \\ 0.5 & 2.5 \end{pmatrix}$$

$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Eigenvectors

Eigenvalues

$$\lambda = 3 \quad \text{or} \quad \lambda = 2$$

If A is an N*N matrix then there are N solutions.

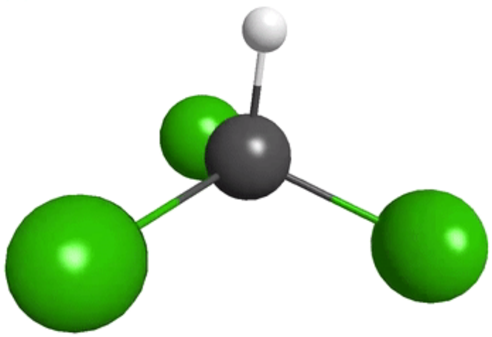
These satisfy

$$\det(A - \lambda I) = 0$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

Vibration in engineering, chemistry, physics



Acoustics, Maxwell, Electromagnetic theory, WiFi

$$\nabla^2 \mathbf{x} = -\omega^2 \mathbf{x}$$

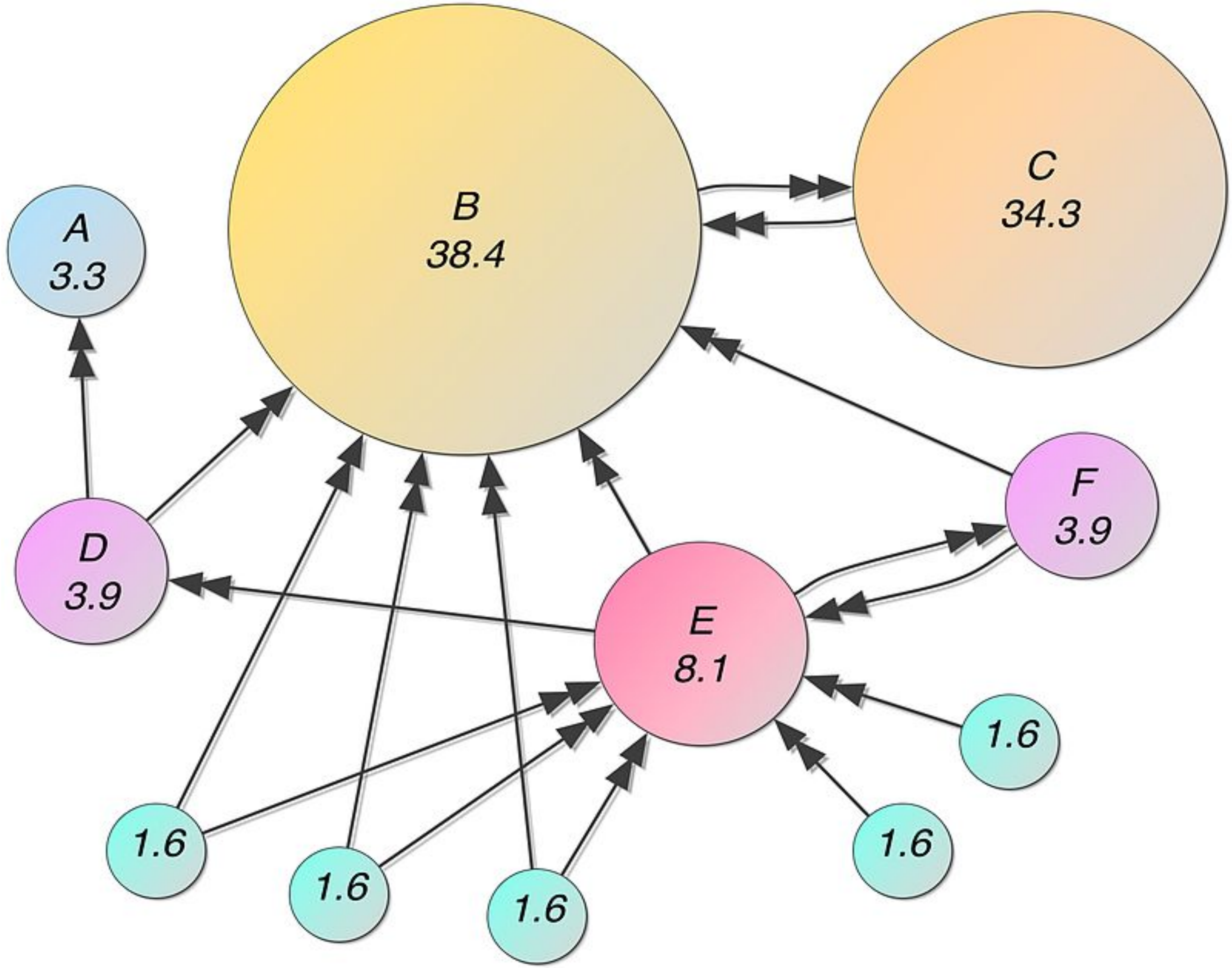
Quantum theory

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\frac{-\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t)\right]\psi$$

PageRank Algorithm

- The ith webpage gets rank R_i
- This page will be pointed to by m other pages.
- These each have ranks R_{ij} and point to N_{ij} other pages
- The ranks satisfy (Damping d = 0.85)

$$R_i = \frac{(1-d)}{N} + d\left(\frac{R_{i1}}{N_{i1}} + \frac{R_{i12}}{N_{i2}} + \ldots + \frac{R_{im}}{N_{im}}\right).$$



Wikipedia

Posed as a matrix eigenvalue equation

$$M\ R = R$$

M: Augmented adjacency matrix HUGE

Eigenvalue of one. Need to find eigenvector R

Use iteration $$R^{n+1} = M\ R^n$$

Works fast and is the basis of Google

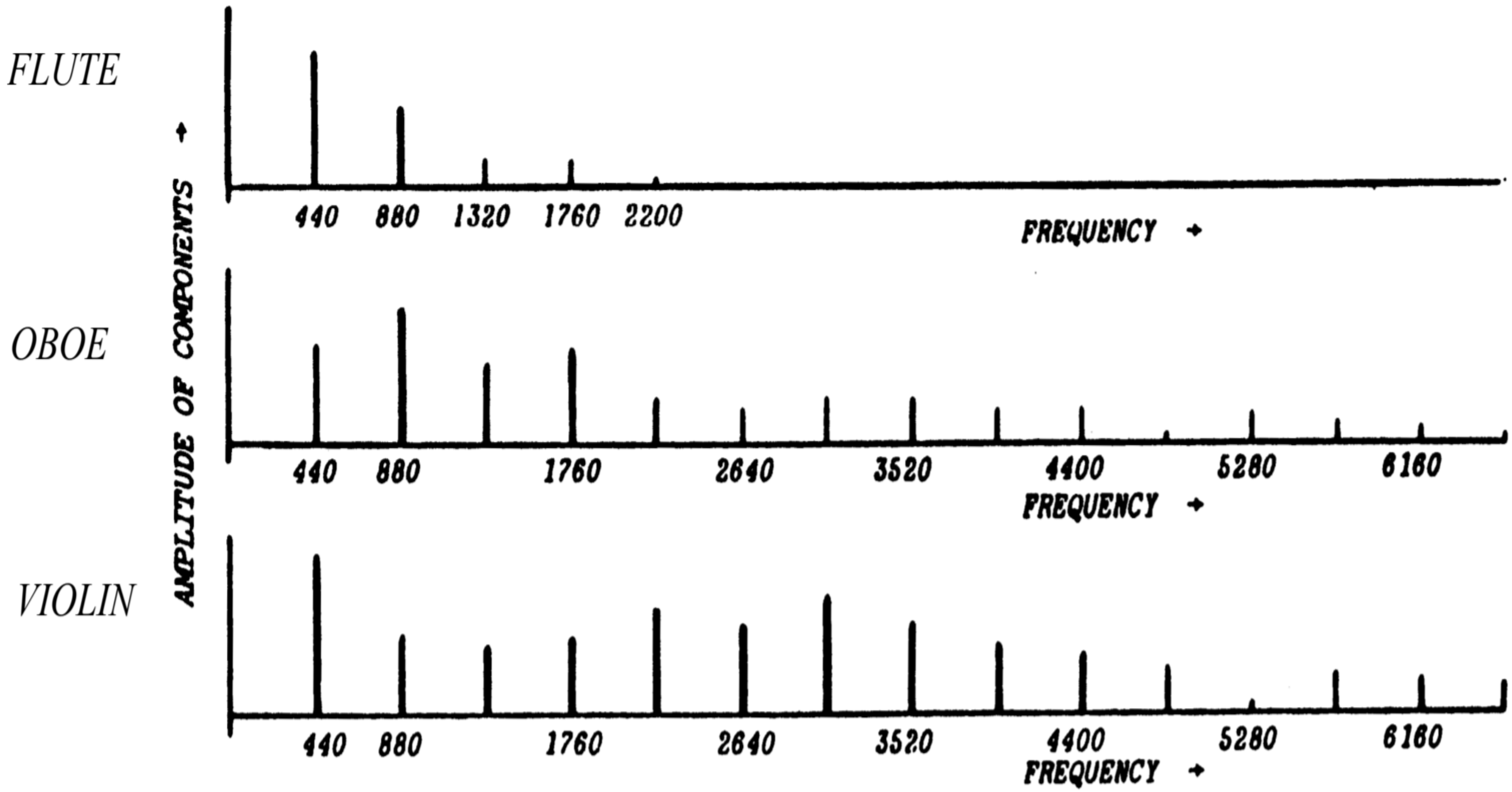
Equation 3: The Fourier Transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt,$$

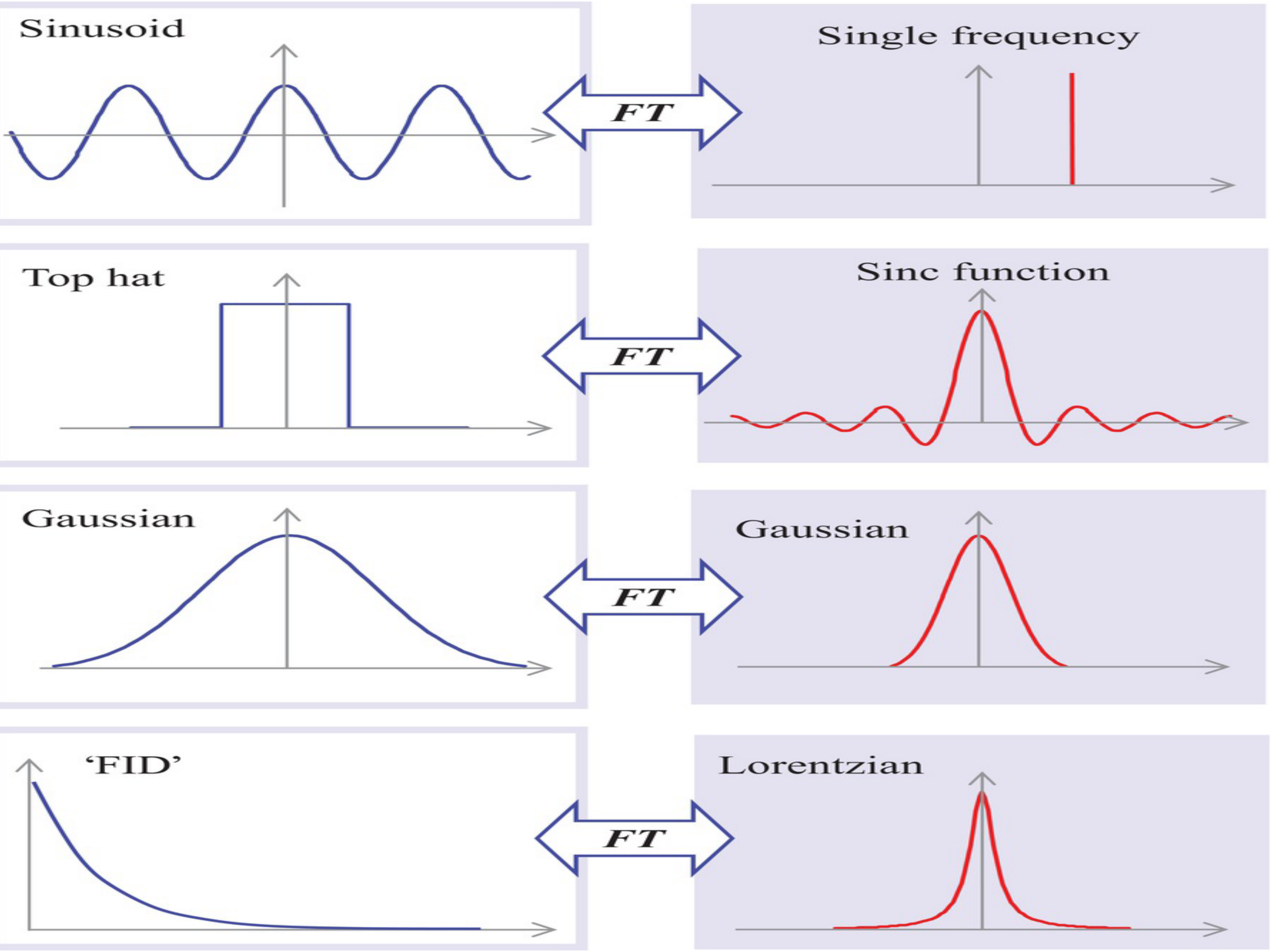
$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \hat{f}(\omega)\, d\omega$$

Decomposing a function into its constituent waves

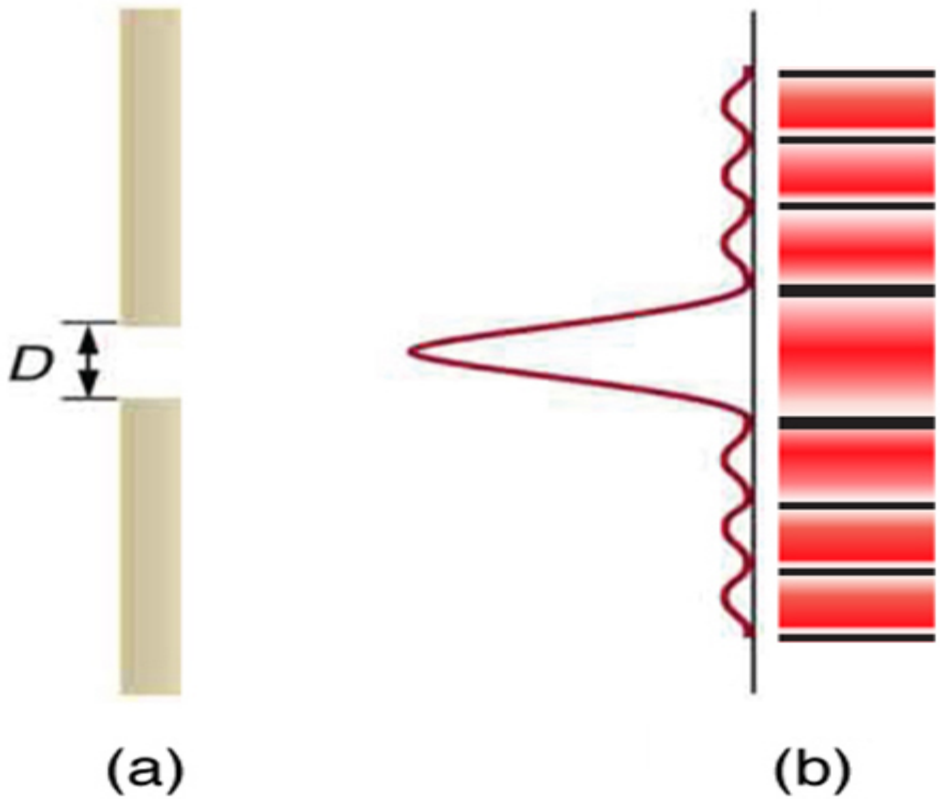
Look at the different harmonics of the notes



Find these harmonics by using the Fourier Transform


Sinusoid
Single frequency
FT
Top hat
Sinc function
FT
Gaussian
Gaussian
FT
'FID'
Lorentzian
FT

Diffraction of light through a slit



Fourier Transform is THE essential tool in optics, telecommunications, image processing, spectral analysis, acoustics, ….

Want to communicate over a channel

Alice → Channel → Bob

$f(t)$ $g(t)$ $h(t)$

$$h(t) = \int_{-\infty}^{\infty} f(t-\tau)\, g(\tau)\, d\tau.$$

Convolution

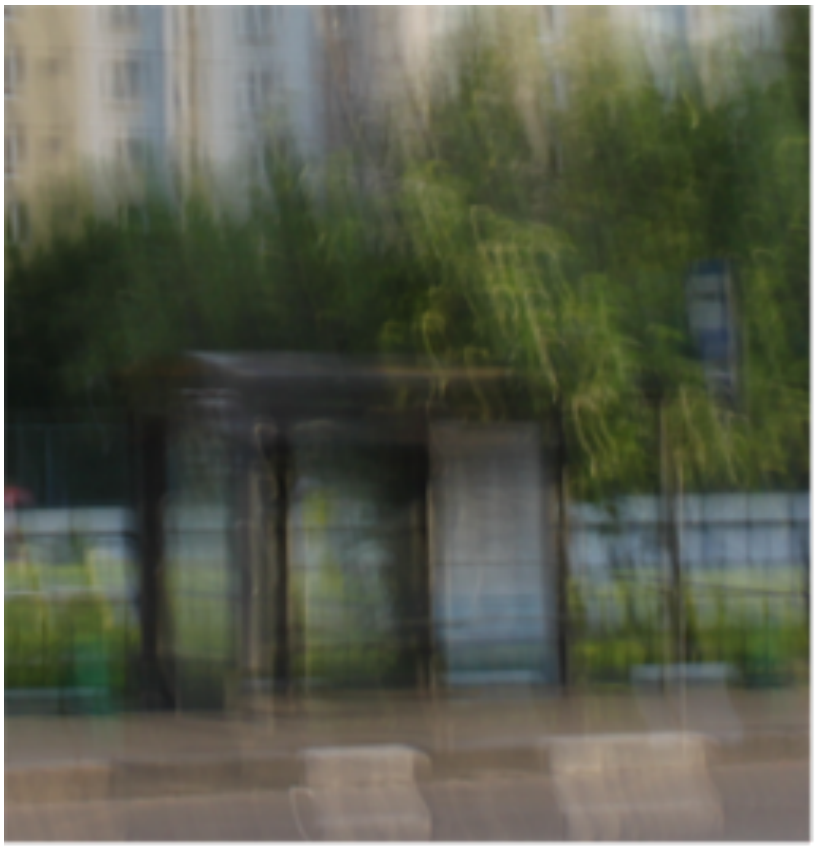
$$h = f * g$$

$$\hat{h} = \hat{f}\,\hat{g}.$$

Convolution theorem

$$\hat{f} = \frac{\hat{h}}{\hat{g}}$$

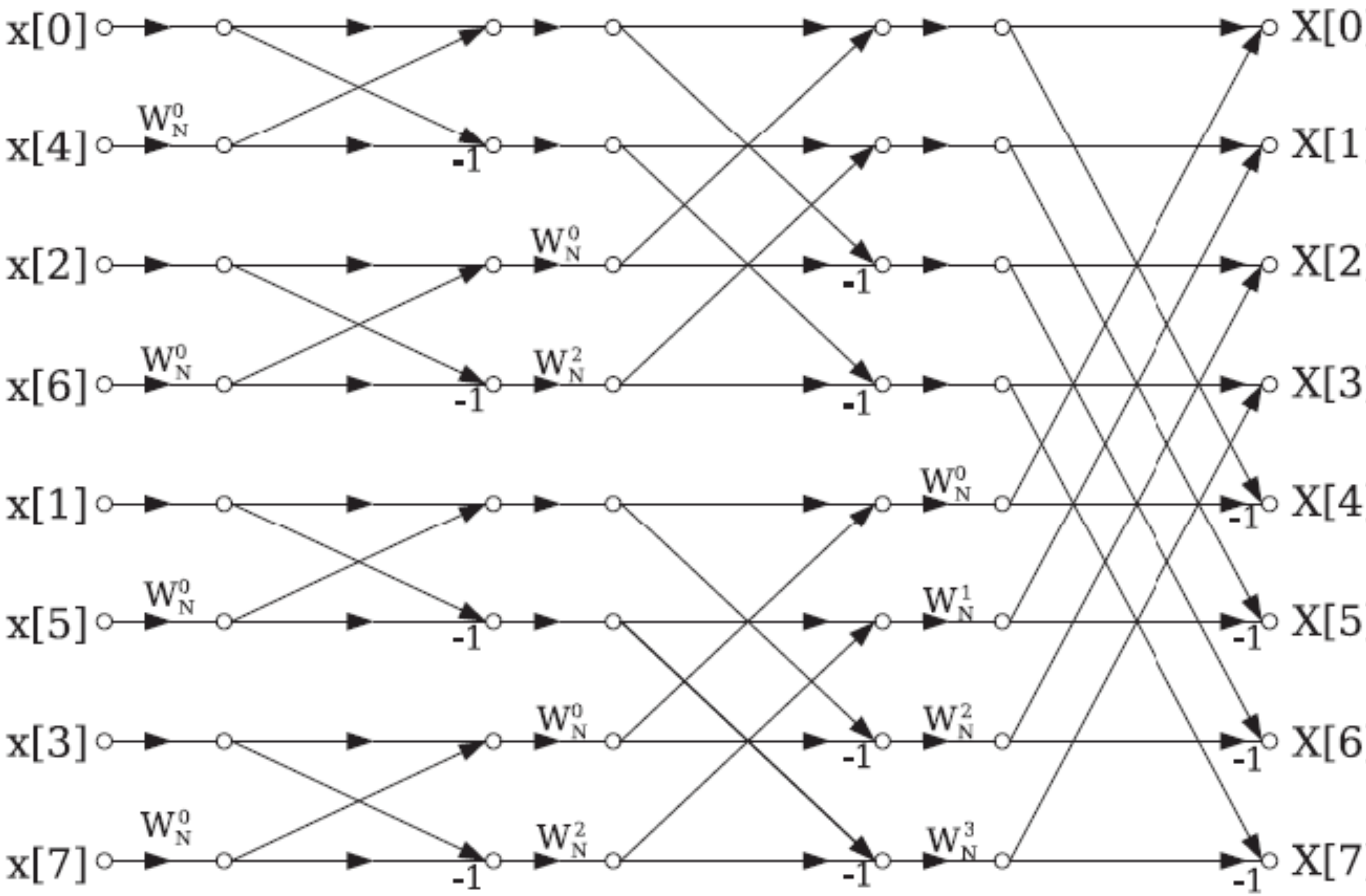
Deblurring

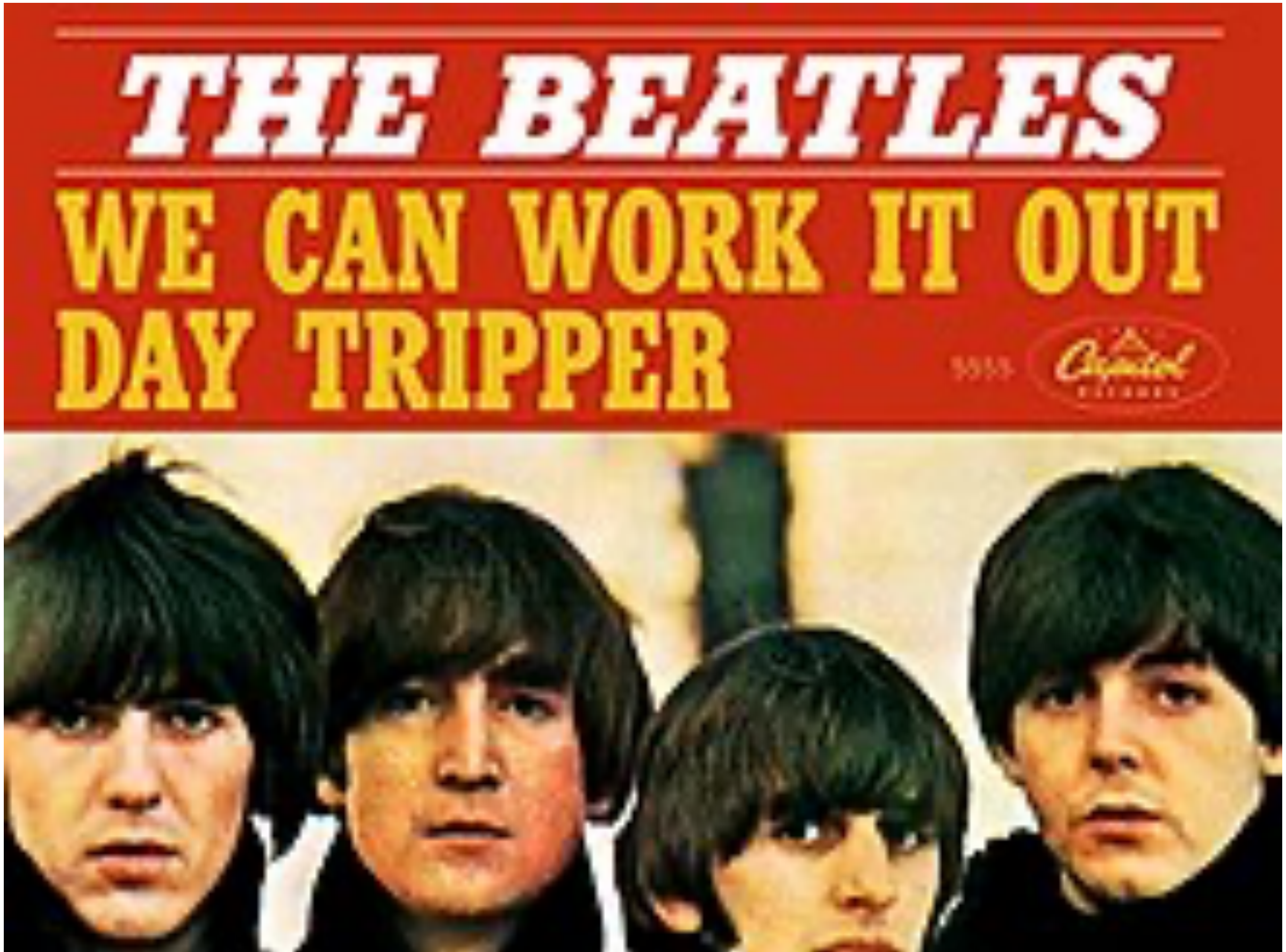

Before


After

MRI Imaging

Evaluate using the FFT Algorithm (1965)




THE BEATLES
WE CAN WORK IT OUT
DAY TRIPPER
Capitol

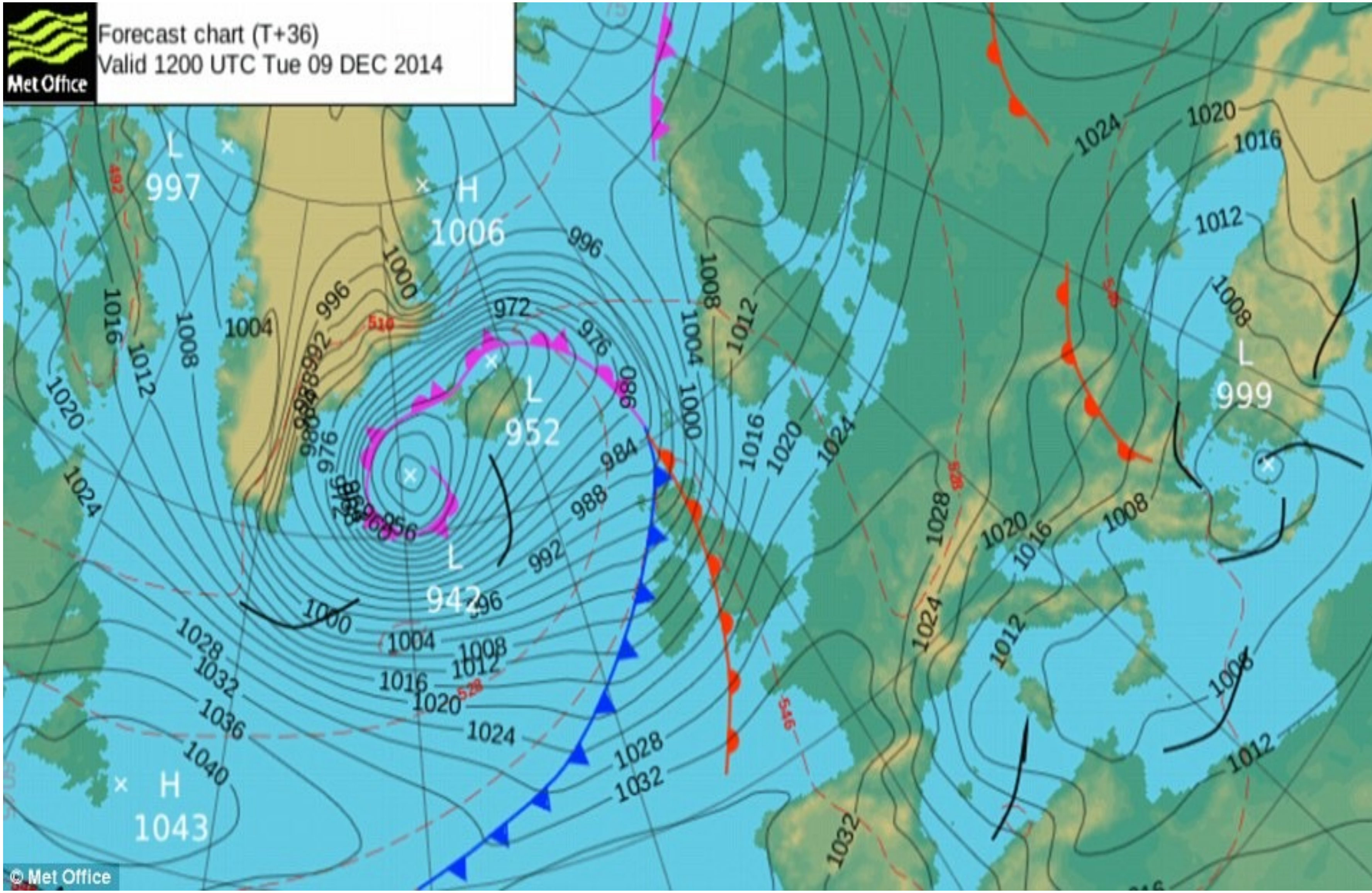
Equation 4: The Navier-Stokes Equations

$$\frac{D\mathbf{u}}{Dt} + 2\mathbf{f} \times \mathbf{u} = -\frac{1}{\rho}\nabla P + \frac{1}{Re}\nabla^2\mathbf{u} - g\mathbf{k},$$

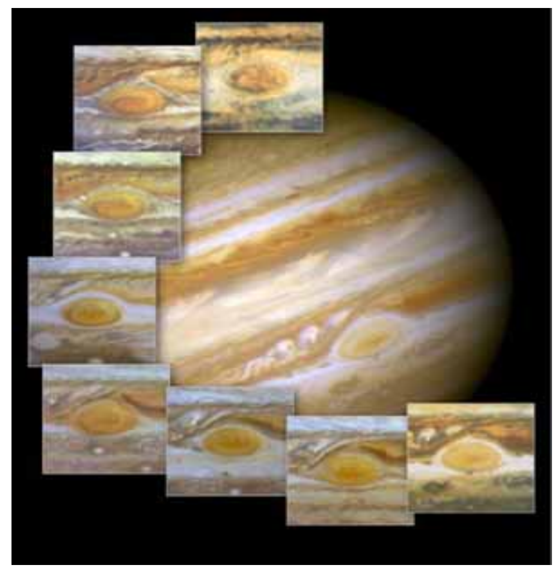
$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\mathbf{u}) = 0.$$

The equations of the weather

The equations for weather and climate



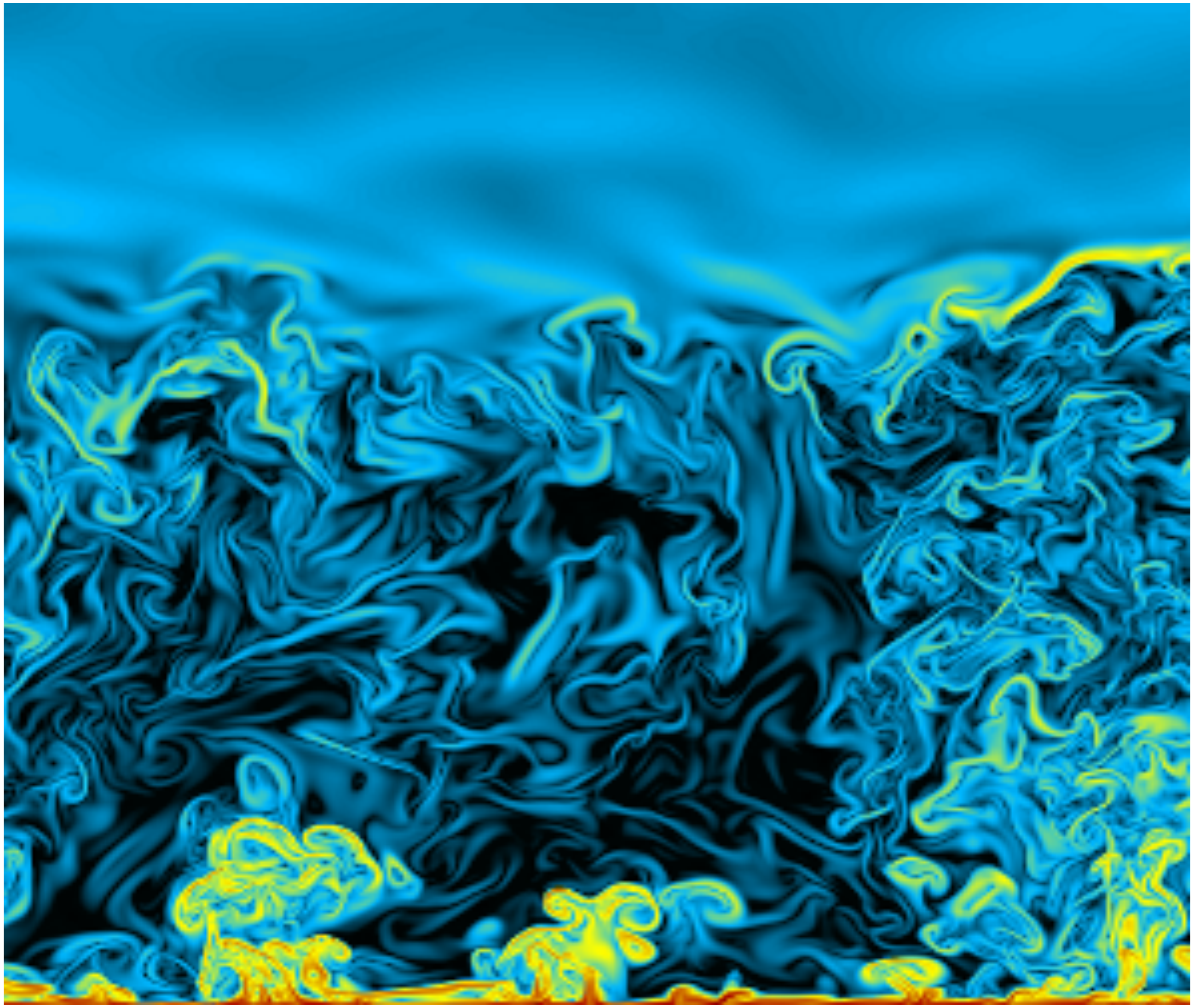


The Navier-Stokes equations:

- Are extremely important
- Are VERY hard to solve, even on a super computer
- Have very complex, and even chaotic solutions
- We don’t even know if they have regular solutions at all!

Turbulence



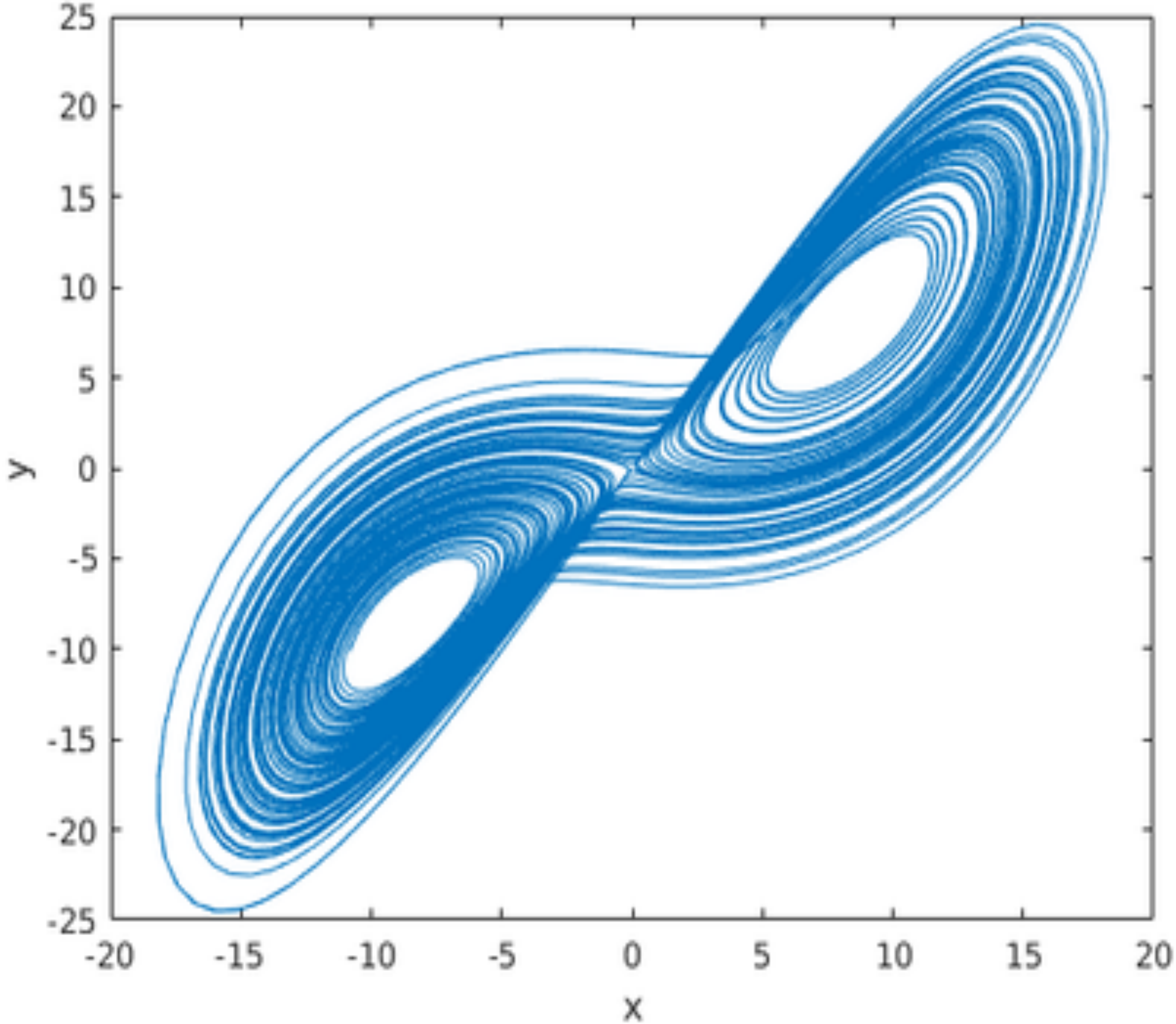
The Lorenz Equations for chaos

$$\frac{dx}{dt} = \sigma(y - x),$$

$$\frac{dy}{dt} = x(\rho - z) - y,$$

$$\frac{dz}{dt} = xy - \beta z.$$

Chaotic strange attractor



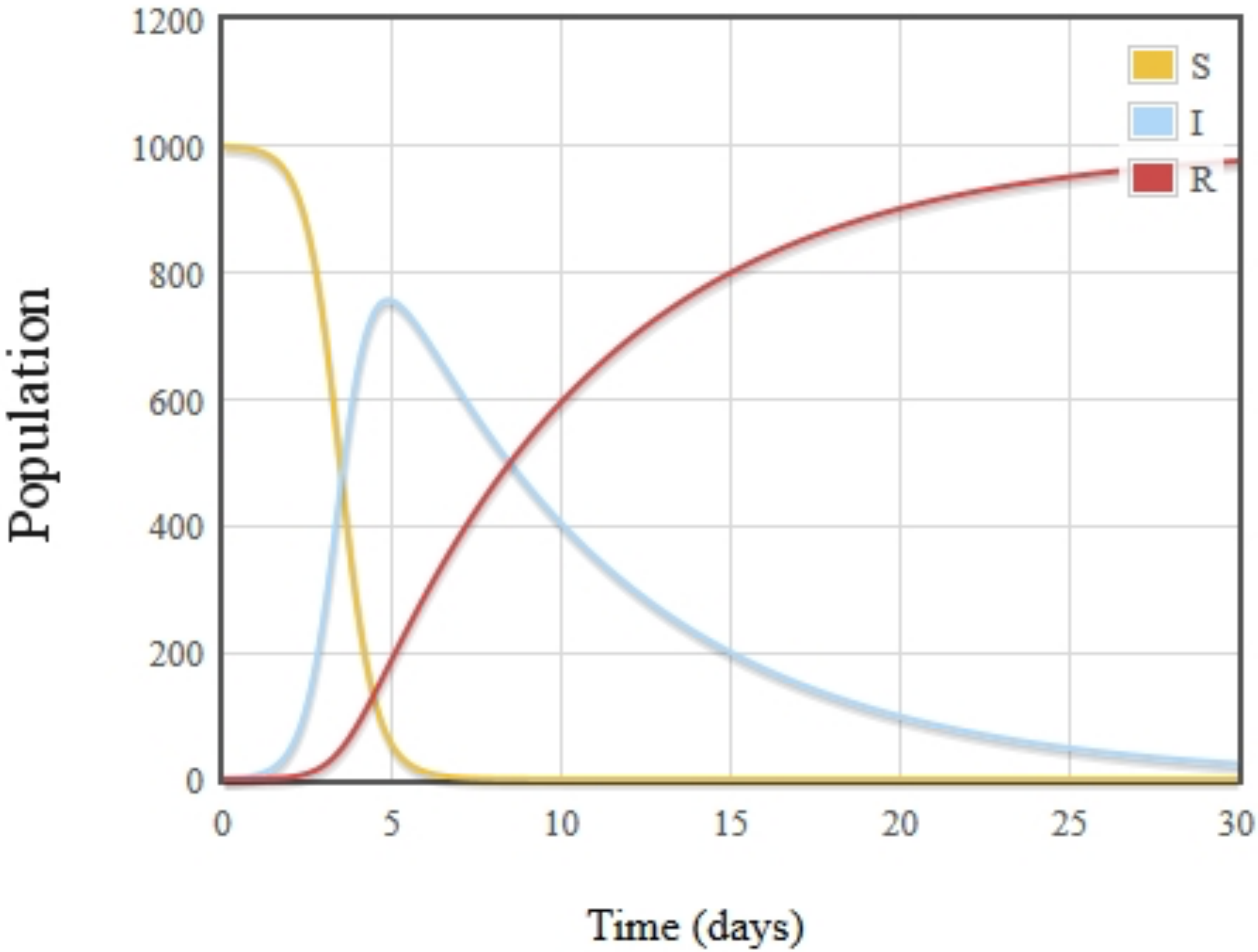
The SIR equations for an epidemic

$$\frac{dS}{dt} = -\frac{\beta I S}{N},$$

$$\frac{dI}{dt} = \frac{\beta I S}{N} - \gamma I,$$

$$\frac{dR}{dt} = \gamma I.$$

Predictions of the SIR model



Equation 5: The Shower Equation

$$\frac{dx}{dt} = -\lambda \, x(t - \tau).$$

This equation explains how we can control things

Can you control a shower if there is a delay of τ between the control and its effect?

$$\frac{dx}{dt} = -\lambda\, x(t-\tau).$$

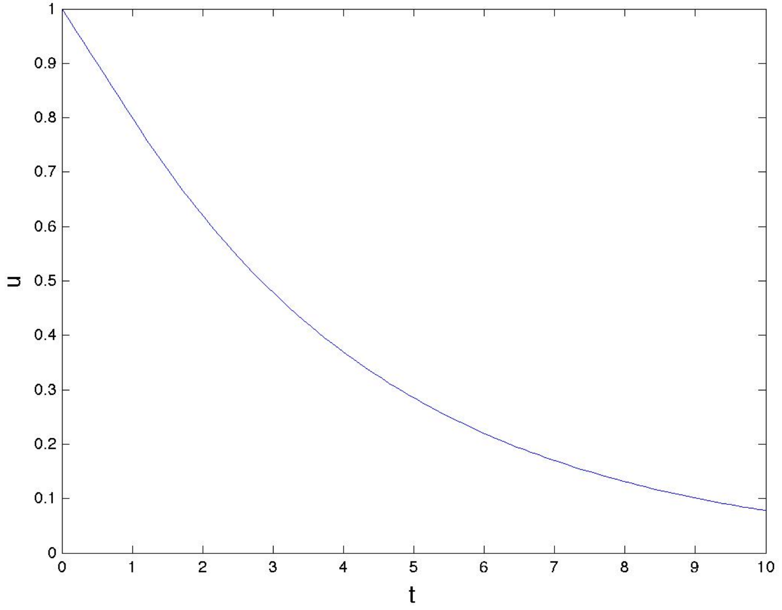
$$x(t) = e^{\alpha t}$$

$$\alpha e^{\alpha t} = -\lambda e^{\alpha(t-\tau)} = -\lambda e^{\alpha t} e^{-\alpha\tau}$$

$$\alpha = -\lambda\, e^{-\alpha\tau}.$$

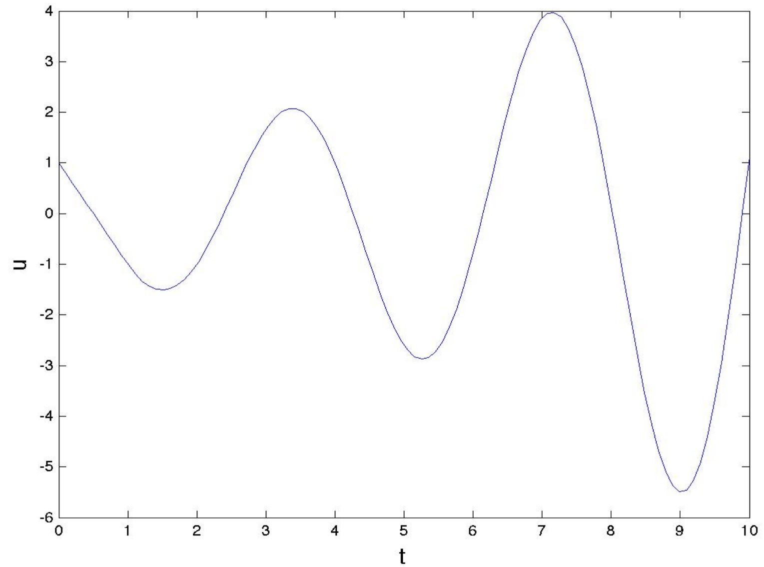
Stable control only if the real part of alpha is negative

$$\lambda\tau < \frac{\pi}{2}$$



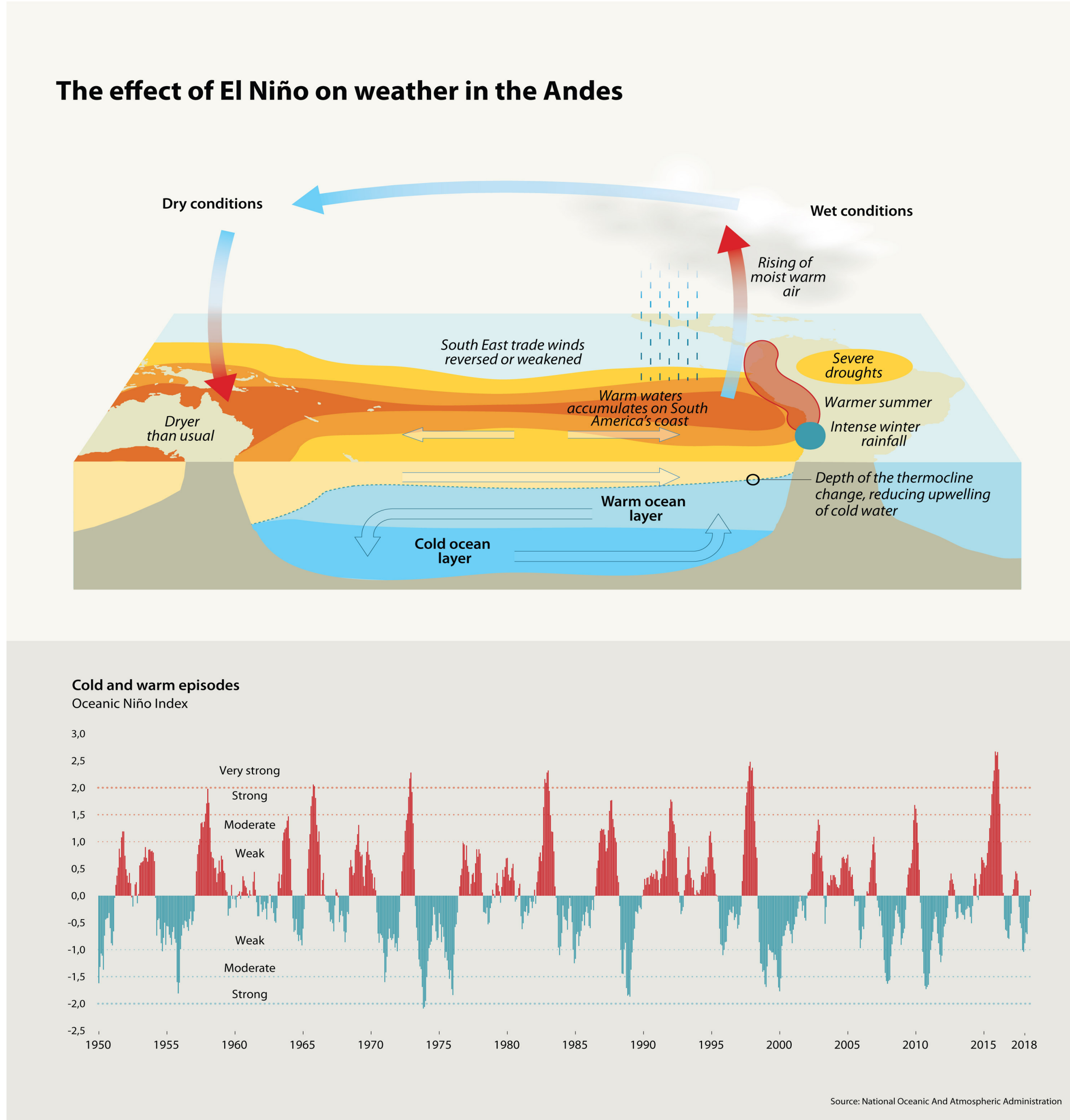
Controlled

$$\lambda\tau > \frac{\pi}{2}$$



Uncontrolled

The El Nino Southern Oscillation ENSO



Delay leads to hard to predict behaviour

Most things we try to control have a delay

This can make it very hard to control them

The shower problem is a good example

So is the impact of responses to the COVID-19 emergency

There has never been a time when an understanding of mathematics is more important to our lives